ON TOPOLOGICAL INSTABILITIES IN FAMILIES OF SEMILINEAR
PARABOLIC PROBLEMS SUBJECT TO NONLINEAR PERTURBATIONS

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Abstract. In this article it is proved that the dynamical properties of a broad class of semi-
linear parabolic problems are sensitive to arbitrarily small but smooth perturbations of the
nonlinear term, when the spatial dimension is either equal to one or two. This topological insta-
bility is shown to result from a local deformation of the global bifurcation diagram associated
with the corresponding elliptic problems. Such a deformation is shown to systematically occur
via the creation of either a multiple-point or a new fold-point on this diagram when an appro-
priate small perturbation is applied to the nonlinear term. This is accomplished for continuous,
locally Lipschitz but not necessarily \(C^1\) nonlinear terms, that prevent in particular the use of
linearization techniques, and for which the family of semigroups may exhibit non-dissipative
properties.

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1. Introduction

Bifurcations of local attractors arising in semilinear parabolic problems and related bifurca-
tions in semilinear elliptic problems, have been thoroughly studied in the literature since the
pioneering works of [Ama76, Rab73, CR73, MM76, Hen81, Sat73, Sat80], and a large portion
of subsequent works has been devoted to the study of qualitative changes occurring within a

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fixed family when a bifurcation parameter is varied; see e.g. [MW05, MW14, HI11, Kie12] and references therein.

Complementarily, perturbed bifurcation problems arising in families of semilinear elliptic equations, have been considered. These problems, in their general formulation, are concerned with the dependence of the global bifurcation diagram to perturbations of the nonlinear term [KK73]. Such a dependence problem is of fundamental importance to understand, for instance, how the multiplicity of solutions of such equations varies as the nonlinearity is subject to small disturbances, or is modified due to model imperfections [BF82, GS79, KK73].

However, this problem has been mainly addressed in the context of two-parameter families of elliptic problems; see e.g. [BCT88a, BCT88b, BRR80, BF82, DPF90, CHMP75, Du00, DL01, KK73, KL99, Lio82, MS80, She80, SW82]. In comparison, the dependence of the global bifurcation diagram with respect to variations in other degrees of freedom such as the “shape” of the nonlinearity, remains largely unexplored; see however [Dan88, Dan08, Hen05, NS94] for the study of related effects of the variation of the domain.

As we will see, the study of perturbed bifurcation problems of semilinear elliptic equations can be naturally related to the study of topological robustness of dynamical properties associated with the corresponding families of semilinear parabolic equations, once the appropriate framework has been set up. The issue is here not only to translate the deformations of the global bifurcation diagram of the elliptic problems into a dynamical language for the parabolic problems, but also to take into consideration the possible discrepancies of regularity that may arise between the weak solutions of the former and the equilibrium semigroup solutions of the latter.

It is the purpose of this article to introduce such a framework that will allow us in particular, to analyze from a topological viewpoint, the perturbation effects of the nonlinear term on the parameterized families of semigroups associated with semilinear parabolic problems of the form

\[
\begin{align*}
\partial_t u - \Delta u &= \lambda g(u), \quad \text{in } \Omega, \\
 u &= 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

given on a bounded and sufficiently smooth domain \(\Omega \subset \mathbb{R}^d\). Our approach allows us to include both dissipative\(^1\) as well non-dissipative cases; the latter cases being commonly encountered when \(g\) is superlinear such as in combustion theory [BE89, CH98, F-K69, Gel63, QS07] or in plasma physics [BB80, Cha57, Tem75], see also [Fil05].

Within this framework, it is then proved that the dynamical properties of a broad class of semilinear parabolic problems turns out to be sensitive to arbitrarily small perturbations of the nonlinear term, when the spatial dimension \(d\) is either equal to one or two.

This is the content of Theorem 3.2 proved below, which constitutes the main result of this article. The proof of this theorem is articulated around a combination of techniques relative to the generation of discontinuities in the minimal branch borrowed from [CEP02], the growth property of the branch of minimal solutions as recalled in Proposition 3.1 below, and a general continuation result from the Leray-Schauder degree theory, formulated as Theorem A.1 below and proved in Appendix A for the sake of completeness. The latter theorem provides conditions from the Leray-Schauder degree theory, under which the existence of an unbounded continuum of steady states for the corresponding family of semilinear elliptic problems,\(^2\) can be ensured.

The proof of Theorem 3.2 provides furthermore the mechanism at the origin of the aforementioned topological instability of the parameterized family of “phase portraits” associated

\[^{1}\text{In the sense that the associated semigroup exhibits a bounded absorbing set; see [Tem97]}
\[^{2}\text{Considered in } (0, \infty) \times E, \text{ where } E \text{ is a Banach space for which the nonlinear elliptic problem } -\Delta u = \lambda g(u), \quad u|_{\partial \Omega} = 0, \text{ is well-posed, for } \lambda \in \Lambda \subset (0, \infty).\]
with (1.1). More precisely, it is shown that such a topological instability comes from a local
deformation of the $\lambda$-bifurcation diagram associated with the corresponding elliptic problems.
This deformation results from the creation of either a multiple-point or a new fold-point on
this diagram when an appropriate small perturbation is applied to the nonlinear term. This
is accomplished for continuous, locally Lipschitz but not necessarily $C^1$ nonlinear terms, that
prevent the use of linearization techniques, and for which the family of semigroups associated
with (1.1) may exhibit non-dissipative properties.

2. A FRAMEWORK FOR THE TOPOLOGICAL ROBUSTNESS OF FAMILIES OF SEMILINEAR
PARABOLIC PROBLEMS

In Section 2.1 that follows, the perturbed Gelfand problem serves as an illustration of perturbed
bifurcation problems arising in families of semilinear elliptic equations regarding the dependence
of the global bifurcation diagram to perturbations of the nonlinear term [KK73]. As mentioned
in the introduction, such a dependence problem is of fundamental importance to understand, for
instance, how the multiplicity of solutions of such equations varies as the nonlinearity is subject
to small disturbances, or is modified due to model imperfections [BF82, GS79, KK73].

We will see in Section 2.3 below, how perturbed bifurcation problems can be naturally related
to the study of topological robustness of the corresponding families of semilinear parabolic
equations. The latter notion of topological robustness is related to the more standard notion
of structural stability encountered for semilinear parabolic problems [HMO02] and recalled in
Section 2.2 below. However as we will see, our approach, based on the notion of topological
equivalence between parameterized families of semigroups such as introduced in Definition 2.2
(see Section 2.3), adopts a more global point of view than the standard notion of structural sta-
bility, while taking into account the possible discrepancies of regularity that may arise between
the (weak) solutions of elliptic problems, on the one hand, and the equilibrium (semigroup)
solutions of the corresponding parabolic problems, on the other.

The framework introduced in Section 2.3 below allows us, furthermore, to deal with semi-
groups for which some trajectories undergo finite-time blow up, or do not exhibit a compact
absorbing set nor even a bounded absorbing set for all solutions$^3$.

2.1. The perturbed Gelfand problem as a motivation. Given a smooth bounded domain
$\Omega \subset \mathbb{R}^d$, the perturbed Gelfand problem, consists of solving the following nonlinear eigenvalue
problem

\begin{equation}
\begin{aligned}
-\Delta u &= \lambda \exp \left( \frac{u}{1+\varepsilon u} \right), \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega,
\end{aligned}
\end{equation}

of unknown $\lambda > 0$, and $u$ in some functional space. We refer to [BE89, Cha57, F-K69, Gel63,
JL73, Tai95, Tai98, QS07] for more details regarding the physical contexts where such a problem
arises.

We first recall how general features regarding the structure of the solution set of (2.1) (param-
eterized by $\lambda$), can be easily derived by application of Theorem A.1, and the theory of semilinear
elliptic equations [Caz06]; and we point out some open questions related to the exact shape of
this solution set when the nonlinearity is varied by changing $\varepsilon$. Such open questions although
not directly addressed in this article, motivate, in part, the results obtained in Theorem 3.2
proved below.

$^3$The solutions $u(t)$ that are well defined in some Hilbert space for all $t > 0$ but are not bounded in time, are
sometimes referred as “grow-up” solutions; see [Ben10].
Let $\alpha \in (0, 1)$ and consider the Hölder spaces $V = C^{2,\alpha}(\Omega)$ and $E = C^{0,\alpha}(\Omega)$. It is well known (see e.g. [GT98, Chapter 6]) that given $f \in E$ and $\lambda \geq 0$, there exists a unique $u \in V$ of the following Poisson problem,

\begin{equation}
-\Delta u = \lambda \exp\left(\frac{f}{1+\varepsilon f}\right), \quad \text{in } \Omega,
\end{equation}

\begin{equation}
\begin{array}{ll}
u = 0, & \text{on } \partial \Omega.
\end{array}
\end{equation}

This allows us to define a solution map $S : E \rightarrow V$ given by $S(f) = u$, where $u \in V$ is the unique solution to (2.2). By composing $S$ with the compact embedding $i : V \rightarrow E$ [GT98] we obtain a map $\tilde{S} := i \circ S : E \rightarrow E$ which is completely continuous.

Define now $G : \mathbb{R}^+ \times E \rightarrow E$ by $G(\lambda, u) = \lambda \tilde{S}(u)$, and consider the equation,

\begin{equation}
G(\lambda, u) := u - G(\lambda, u) = 0_E.
\end{equation}

The mapping $G$ is a completely continuous perturbation of the identity and solutions of the equation $G(\lambda, u) = 0$ correspond to solutions of (2.1). For any neighborhood $U \subset X$ of $0_E$, the function $u = 0$ is the unique solution to (2.3) with $\lambda = 0$. Moreover,

$$\deg(G(0, \cdot), U, 0_E) = \deg(I, U, 0_E) = 1,$$

and therefore from Theorem A.1 (see Appendix A), there exists a global curve of nontrivial solutions which emanates from $(0, 0)$. Here $\deg(G(0, \cdot), U, 0_E)$ stands for the classical Leray-Schauder degree of $G(0, \cdot)$ with respect to $U$ and $0_E$; see e.g. [Dei85, Nir01]. From the maximum principle these solutions are positive in $\Omega$. Since $u = 0$ is the unique solution for $\lambda = 0$ (up to a multiplicative constant), the corresponding continuum of solutions is unbounded in $(0, \infty) \times E$.

From [Lio82, Theorem 2.3], it is known that there exists a minimal positive solution of (2.1) for all $\lambda > 0$; cf. also Proposition 3.1 below. Furthermore, there exists $\lambda_0^\sharp$ such that for every $\lambda \geq \lambda_0^\sharp$, there exists only one positive solution of (2.1), $u_\lambda$ (cf. [CS84]), and that the branch $\lambda \mapsto u_\lambda$ is increasing; see [Ama76] and see Proposition 3.1 below.

For $\lambda$ small enough, i.e. when $0 < \lambda \leq \lambda_0^\sharp$ for some $\lambda_0^\sharp > 0$, it can be proved that the same conclusions about the uniqueness of positive solutions, as well as about the monotony of the corresponding branch, hold. The problem is then to know what happens for $\lambda \in (\lambda_0^\sharp, \lambda_0^\sharp)$. Theorem A.1 may give some clues in that respect. For instance, since Theorem A.1 ensures that the branch of solutions is a continuum, in case where the existence of three solutions for some $\lambda_0^\sharp \in (\lambda_0^\sharp, \lambda_0^\sharp)$ is guaranteed, then such a continuum is necessarily $S$-like shaped, with multiple turning points, not necessarily reduced to two turning points.

The determination of the exact shape of this continuum, for general domains, is however a challenging problem. For instance it is known that for $\varepsilon \geq 1/4$, the problem (2.1) has a unique positive solution for every $\lambda > 0$ whatever the spatial dimension $d$ is, the branch of solutions being a monotone function of $\lambda$; see e.g. [BIS81, CS84]. However, if $d = 2$ and $\Omega$ is the unit open ball of $\mathbb{R}^2$, then it has been proved in [DL01] that there exists $\varepsilon^\star > 0$ such that for $0 < \varepsilon < \varepsilon^\star$ this continuum is exactly $S$-shaped — i.e. with exactly two turning points — when represented in a “$(\lambda, \| \cdot \|_{\infty})$-plane” classically used for representing bifurcation diagrams such as arising in elliptic problems for domains with radial symmetries [JL73]. The global bifurcation curve can become however more complicated than $S$-shaped, when $\Omega$ is the unit ball in higher dimension; see [Du00] for $3 \leq d \leq 9$.

In the case $d = 1$, a lower bound of the critical value $\varepsilon^\star > 0$, for which for all $0 < \varepsilon < \varepsilon^\star$ the continuum is exactly $S$-shaped, has been derived in [KL99]. It ensures in particular that $\varepsilon^\star \geq \frac{1}{135}$ when $\Omega = (-1, 1)$, which gives a rather sharp bound of $\varepsilon^\star$ in that case, since $\varepsilon^\star \leq \frac{1}{4}$ from the general results of [BIS81, CS84]. Numerical methods with guaranteed accuracy to
enclose a double turning point [Min04] strongly suggest that this theoretical bound can be further improved. Based on such numerical methods and the aforementioned theoretical results, it can be reasonably conjectured that in dimension $d = 1$ and $\Omega = (-1, 1)$, the $\lambda$-bifurcation diagram\(^4\) does not present any turning point (monotone branch) when $\varepsilon > 1/4$, whereas once $\varepsilon < 1/4$, an $S$-shaped bifurcation takes place. We observe here that a continuous change in the parameter $\varepsilon$ can lead to a qualitative change of its $\lambda$-bifurcation diagram on its whole: from a monotone curve to an $S$-shaped curve as $\varepsilon$ crosses $1/4$ from above. It is thus reasonable to conjecture that $\varepsilon^* = 1/4$, for $d = 1$.

From the numerical results of [Min04], it can be inferred that $\varepsilon^* \in (0.238, 0.2396]$ if $d = 3$ and $\Omega$ is the unit ball; emphasizing the dependence of such a critical value on the dimension of the physical space.

**Remark 2.1.** It should be noticed that the results of [Du00] for two-dimensional balls, combined with a domain perturbation technique due to [Dan88], implies that, even in dimension 2, if $\Omega$ is the union of several balls touched slightly, then the number of positive solutions of (2.1) may be greater than three for some values of $\lambda$. This feature indicates that the $\lambda$-bifurcation diagram is not necessarily $S$-shaped, even in dimension 2.

### 2.2. Structural stability for dissipative semilinear parabolic problems

The qualitative changes, when $\varepsilon$ is varied, of the global $\lambda$-bifurcation diagram recalled above for the perturbed Gelfand problem is reminiscent with the so-called *cusp bifurcation* observed in two-parameter families of autonomous ordinary differential equations (ODEs) [Kuz04].

Indeed, if we consider the paradigmatic example, $\dot{x} = \beta_1 + \beta_2 x - x^3$, where $x \in \mathbb{R}$, $\beta = (\beta_1, \beta_2) \in \mathbb{R}^2$, we can easily exhibit two bifurcation curves in the $(\beta_1, \beta_2)$-plane. These bifurcation curves are given by $\gamma_{+/-} = \{(\beta_1, \beta_2) : \beta_1 = \pm \frac{1}{3\sqrt{3}} \beta_2^{3/2}, \beta_2 > 0\}$, on which saddle-node bifurcation occur, i.e. collision and disappearance of two equilibria [Kuz04]. This two curves divide the parameter plane into two regions: inside the “dead-end” formed by $\gamma_+$ and $\gamma_-$, there are three steady states, two stable and one unstable, and outside this corner, there is a single steady state, which is stable. A crossing of the cusp point $\beta = (0, 0)$ from outside the “dead-end,” leads to an unfolding of singularities [Arn81, Arn83, CT97, GS85] which consists more exactly to an unfolding of three steady states from a single stable equilibrium; see also [Kuz04].

The qualitative changes described at the end of the previous section may be therefore interpreted in that terms; see also [MN07, Fig. 1]. Singularity theory is a natural framework to study the effects on the bifurcation diagram of small perturbations or imperfections to a given (static) model [GS79, GS85]. In that spirit, geometric connections between a double turning point and a cusp point have been discussed for certain nonlinear elliptic problems in [BCT88, BF82, MS80, SW82], but a general treatment of the effects of arbitrary perturbations on bifurcation diagrams arising for such problems has not been fully achieved, especially when the perturbations are not necessarily smooth; see however [Dan08, Hen05] for related problems.

In that respect, it is tempting to describe the aforementioned qualitative changes in terms of structural instability such as defined in classical dynamical systems theory [AM87, Arn83, Sma67]. Nevertheless, as we will see in Section 2.3, such topological ideas have to be recast into a formalism which takes into account the functional settings for which the parabolic and corresponding elliptic problems are considered; see Definitions 2.1, 2.2 and 2.5 below. This formalism will be particularly suitable for problems such as arising in combustion theory for which the associated semigroups are not necessarily dissipative. To better appreciate this distinction with

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\(^4\)By $\lambda$-bifurcation diagram we mean the bifurcation diagram obtained when $\lambda$ is varied and $\varepsilon$ is fixed.
the standard theory, we recall briefly below the notion of structural stability such as encountered for dissipative infinite-dimensional systems.

The concept of structural stability has been originally introduced for finite-dimensional $C^1$-vector fields in [AP37]. A system is said to be structurally stable if roughly speaking a small, sufficiently smooth\(^5\) perturbation of this system preserves its dynamics up to a homeomorphism, i.e. up to a bijective continuous change of variables (with continuous inverse) that transforms the phase portrait of this system into that of the perturbed system; see e.g. [AM87, Arn83, KH97, New11, Sma67]. The existence of such an homeomorphism is related to the study of conjugacy or topological equivalence problems; see e.g. [AM87, Arn83, CGVR06, CR13, KH97, Sma67] and references therein.

Structural stability has also been investigated for various types of infinite-dimensional dynamical systems, mainly dissipative. As a rule of thumb for such dynamical systems, one investigates structural stability of the semiflow restricted to a compact invariant set, usually the global attractor, rather than the flow in the original state space [HMO02, Definition 1.0.1]; an exception can be found in e.g. [Lu] where the author considered the semiflow in a neighborhood of the global attractor.

In the context of reaction-diffusion problems, the problem of structural stability is concerned with,

\[(2.4) \begin{cases} \partial_t u - \Delta u = g(u), & \text{in } \Omega, \ g \in C^1(\mathbb{R}, \mathbb{R}), \\ u|_{\partial\Omega} = 0, \end{cases}\]

that is assumed to generate a semigroup \(\{S(t)\}_{t \geq 0}\) for which a global attractor \(A_g\), in some Banach space \(X\), exists [BP97, FR99, HMO02, Lu].

Within this context, the structural stability problem may be formulated as the existence problem of an homeomorphism \(H : A_g \rightarrow \hat{A}_g\) for arbitrarily small perturbations \(\hat{g}\) of \(g\) in some topology \(T\) of \(C^1(\mathbb{R}, \mathbb{R})\), that aims to satisfy the following properties

\[(2.5a) \quad \hat{A}_g \text{ is a global attractor in } X \text{ of } \{\hat{S}(t)\}_{t \in \mathbb{R}^+}, \text{ and }\]

\[(2.5b) \quad \forall \ t \in \mathbb{R}, \forall \phi \in A_g, \ H(S(t)\phi) = \hat{S}(t)H(\phi),\]

where \(\{\hat{S}(t)\}_{t \geq 0}\) denotes the semigroup generated by

\[u_t - \Delta u = \hat{g}(u), \ u|_{\partial\Omega} = 0.\]

The topology \(T\) may be chosen to be the compact-open topology or the finer topology of Whitney; see [Hir76] for general definitions of these topologies; and see [BP97] for questions regarding the genericity of structurally stable reaction-diffusion problems of type (2.4), making use of the Whitney topology. Note that in (2.5b)\(^6\), the restriction of the dynamics to the global attractor, allows us to consider backward trajectories onto the global attractor giving rise to genuine flows onto the global attractor; see e.g. [FR99, Rob01].

A necessary condition in order that a parabolic equation, generating a semigroup, possesses a global attractor in a Banach \(X\), is to satisfy a dissipation property, i.e. to verify the existence of an absorbing ball in \(X\) for this semigroup (see e.g. [MWZ02, Theorem 3.8]), which in particular

\(^5\)typically \(C^1\).

\(^6\) Note that (2.5b) may be substituted by the more general condition requiring that for all \(t \in \mathbb{R}\), and for all \(\phi \in A_g\), \(H(S(t)\phi) = \hat{S}(\gamma(t, \phi))H(\phi)\), with \(\gamma : \mathbb{R} \times A_g \rightarrow \mathbb{R}\) an increasing and continuous function of the first variable. Although this condition is often encountered in the literature, its use is not particularly required when with the questions considered in the present article; see Remark 2.4 below.
prevents any blow-up in finite or infinite time\(^7\) (at least in \(X\)); see [Hal88, Rob01, SY02, Tem97] for classical conditions on \(g\) ensuring the existence of such dissipative semigroups in cases where \(X\) is a Hilbert space. In case of structural stability, it is worthwhile to note that the set of equilibria in \(X\) of \(\{S(t)\}_{t\in\mathbb{R}^+}\) is in one-to-one correspondence with the set of equilibria of \(\{\hat{S}(t)\}_{t\in\mathbb{R}^+}\).

In specific applications, families of semigroups \(\mathcal{S} = \{S_\lambda\}_{\lambda \in \Lambda}\) depending upon a parameter \(\lambda\) in some metric space \(\Lambda\), and possessing a global attractor \(\mathcal{A}_\lambda\) for each \(\lambda \in \Lambda\), may arise. In such a context, the notion of \(\mathcal{A}\)-stability has been introduced [HMO02, Definition 1.0.2] to study in particular the loss of structural stability within the family when the control parameter \(\lambda\) is varied, \(i.e.\) to study the occurrence of \(S_{\lambda_c}\) at some critical value \(\lambda_c\) such that for any neighborhood \(U\) of \(\lambda_c\), there exists \(S_\mu \in \mathcal{S}\) which is non equivalent in the sense of (2.5) to \(S_{\lambda_c}\) for some \(\mu \in U\). The loss of \(\mathcal{A}\)-stability refers therefore to a notion of bifurcation of global attractors\(^8\). However, the underlying assumption implying that the related semigroups generated by a family of semilinear parabolic problems are dissipative may be viewed as too restrictive, since in many applications — such as in combustion theory — blow-up in finite or infinite time may occur for certain trajectories; see [BE89, Ben10, CH98, F-K69, Fil05, QS07].

2.3. Topological robustness for general families of semilinear parabolic problems. To deal with the problem of topological equivalence between families of semigroups which are not necessarily dissipative, we start by introducing several intermediate concepts that we illustrate on some examples borrowed from the literature.

Let us first consider a parameterized family \(\mathfrak{F}_f := \{f_\lambda\}_{\lambda \in \Lambda}\) of functions \(I \to \mathbb{R}\), where \(\Lambda\) is a metric space, and \(I\) is an unbounded interval of \(\mathbb{R}\). We are concerned with the associated parameterized family of semilinear parabolic problems,

\[
(P_{f_\lambda}) \quad \begin{cases} 
\partial_t u - \Delta u = f_\lambda(u), & \text{in } \Omega, \\
\quad u = 0, & \text{on } \partial\Omega,
\end{cases}
\]

where \(\Omega\) is an open bounded subset of \(\mathbb{R}^d\), with additional regularity assumptions on its boundary and \(f_\lambda\) when needed.

In general, these problems may generate a family of semigroups acting on a phase space \(X\) that does not necessarily agree with the functional space \(Y\) on which the (weak) solutions of

\[
(2.6) \quad \begin{cases} 
-\Delta u = f_\lambda(u), & \text{in } \Omega, \\
\quad u = 0, & \text{on } \partial\Omega,
\end{cases}
\]

exist. As we will see in Example 2.1 below, such situations arise when weak solutions to (2.6) do not necessarily correspond to equilibria of the semigroup associated with \((P_{f_\lambda})\). These considerations lead us naturally to introduce the following definition.

**Definition 2.1.** Let \(\Lambda\) be a metric space. Let \(Y\) be a Banach space and \(\Omega\) be an open bounded subset of \(\mathbb{R}^d\), such that (2.6) makes sense in \(\Omega\).

Given a Banach space \(X\), a family \(\mathfrak{F}_f := \{f_\lambda\}_{\lambda \in \Lambda^*}\) will be said to be \((X;Y)\)-compatible relatively to \(\Lambda^* \subset \Lambda\) and \(\Omega\), if there exists a subset \(\Lambda^* \subset \Lambda\), such that for all \(\lambda \in \Lambda^*\) the following properties are satisfied:

(i) There exists a nonempty subset \(D(f_\lambda) \subset X\) such that \((P_{f_\lambda})\) generates a semigroup \(\{S_\lambda(t)\}_{t \geq 0}\) on \(D(f_\lambda)\).

(ii) The set \(\mathcal{E}_{f_\lambda} := \{u \in Y : -\Delta u = f_\lambda(u), \ u|_{\partial\Omega} = 0\}\) is non-empty.

\(^7\)Note that some authors, \(e.g.\) [Hal88], have referred to dynamical systems with this property as having bounded dissipation.

\(^8\)See [MW05] for a notion of bifurcation of local attractors.
(iii) The set $\mathcal{E}_{f\lambda}$ of equilibria of $\{S_{\lambda}(t)\}_{t \geq 0}$, satisfies
$$
\mathcal{E}_{f\lambda} := \{ \phi \in D(f_{\lambda}) : S_{\lambda}(t)\phi = \phi, \ \forall \ t \geq 0 \} = \mathcal{E}_{f\lambda}.
$$
If instead of (iii),
$$
(2.7) \quad \overline{\mathcal{E}_{f\lambda}}^X = \mathcal{E}_{f\lambda}, \text{ with } \mathcal{E}_{f\lambda} \subsetneq \mathcal{E}_{f\lambda},
$$
then $\mathcal{E}_{f\lambda}$ will be said to be weakly $(X; Y)$-compatible relatively to $\Lambda^* \subset \Lambda$ and $\Omega$.

**Remark 2.2.** When the domain $\Omega$ is clear from the context, we will simply say that a family of functions is $(X; Y)$-compatible without referring to $\Omega$. We will also often say that the family of elliptic problems (2.6) is $(X; Y)$-compatible, when the corresponding family of function $\{f_{\lambda}\}$ is $(X; Y)$-compatible.

We first provide an example of a family of superlinear elliptic problems that is not $(C^1(\Omega); H^1_0(\Omega))$-compatible, but only weakly $(C^1(\Omega); H^1_0(\Omega))$-compatible.

**Example 2.1.** It may happen that $\mathcal{E}_{f\lambda} \neq \mathcal{E}_{f\lambda}$ for some $\lambda \in \Lambda^*$. The Gelfand problem [Gel63, Fuj69],
$$
(2.8) \quad -\Delta u = \lambda e^u, \ u|_{\partial B_1(0)} = 0,
$$
where $B_1(0)$ is a unit ball of $\mathbb{R}^d$ with $3 \leq d \leq 9$, is an illustrative example of such a distinction that may arise between the set of equilibrium points and the set of steady states, depending on the functional setting adopted.

In that respect, let us first recall that for $Y = H^1_0(B_1(0))$ there exists $\lambda^* > 0$ such that for $\lambda > \lambda^*$ there is no solution to (2.8), even in a very weak sense [BCMR96], whereas for $\lambda \in [0, \lambda^*]$ there exists at least a solution (in $Y$) so that $\mathcal{E}_f \neq \emptyset$; see [BV97] and Proposition 3.1 below.

In what follows we denote by $A_p$ the (closed) Laplace operator considered as an unbounded operator on $L^p(B_1(0))$ under Dirichlet conditions, with domain
$$
D(A_p) = W^{2,p}(B_1(0)) \cap W^{1,p}_0(B_1(0));
$$
see [Paz83, Sect. 7.3].

Let us now take $\Lambda^* \subset [0, \lambda^*]$ and let us choose $X$ to be the following subspace constituted by radial functions
$$
(2.9) \quad X := \{ \varphi(r) : \varphi \in D(A^\beta_p) \},
$$
where $D(A^\beta_p)$ denotes the domain of $A^\beta_p$, the fractional power of $A_p$, where $0 < \beta \leq 1$; see e.g. [Paz83, Sect. 2.6] and [Hen81, Sect. 1.4].

For $p > d$ and $1 > \beta > (d + p)/(2p)$, it is known that $D(A^\beta_p)$ is compactly embedded in $C^1(B_1(0))$ [Hen81, Thm. 1.6.1], and thus $X \hookrightarrow C^1(B_1(0))$. Then for any $\lambda \in [0, \lambda^*]$ and for such a choice of $p$ and $\beta$, it can be proved that the parabolic problem $(\mathcal{P}_{f\lambda})$ is well posed in $X$ [CH98, SY02, Lun95] with $f_{\lambda}(x) = \lambda \exp(x)$.

As a consequence, by considering
$$
(2.10) \quad D(f_{\lambda}) := \{ u_0 \in X : u_\lambda(t; u_0) \text{ exists for all } t > 0, \text{ and } \sup_{t > 0} \| A^\beta_p u_\lambda(t; u_0) \|_p < \infty \},
$$
a nonlinear semigroup $\{S_{\lambda}(t)\}_{t \geq 0}$ on $D(f_{\lambda})$ can be defined as follows
$$
(2.11) \quad S_{\lambda}(t)u_0 := u_\lambda(t; u_0), \ t \geq 0, u_0 \in D(f_{\lambda}),
$$
where $u_\lambda(t; u_0)$ denotes the solution of $(\mathcal{P}_{f\lambda})$ emanating from $u_0$ at $t = 0$. 

However the property (iii) of Definition 2.1 is not verified here. Indeed, for \( \lambda = \lambda^* = 2(d-2) \in (0, \lambda^*) \) there exists in \( H^1_0(B_1(0)) \) an unbounded solution of the Gelfand problem (2.8) — in the weak sense of [BCMR96] — given by
\[
u_{\lambda}(x) := -2 \log \|x\|,
\]
see [BV97].

This solution does not belong to \( D(f_{\lambda}) \subset X \subset C^1(\overline{B_1(0)}) \) and in particular to \( \mathcal{E}_{f_{\lambda^*}} \), the set of equilibria of \( S_{\lambda}(t) \) in \( D(f_{\lambda}) \) given by (2.10).

Therefore the family
\[
(2.12) \quad \mathfrak{F}_{\exp} = \{ x \mapsto \lambda e^x, \lambda \geq 0, \lambda \in [0, \lambda^*] \},
\]
is not \( (C^1(\overline{B_1(0)}); H^1_0(\overline{B_1(0)})) \)-compatible relatively to \([0, \lambda^*]\) where \( B_1(0) \) is the unit ball of \( \mathbb{R}^d \), for \( 3 \leq d \leq 9 \).

Nevertheless this family is weakly \( (C^1(\overline{B_1(0)}); H^1_0(\overline{B_1(0)})) \)-compatible relatively to \([0, \lambda^*]\), in the sense of Definition 2.1. This property results from the fact that the singular steady state \( u_{\lambda^*} \) can be approximated by a sequence of equilibria in \( X \) for the relevant topology [BV97, JL73], so that in particular condition (2.7) is verified.

The following proposition identifies a broad class of families of sublinear elliptic problems which are \( (C^0_0,2\alpha([0,1]);C^2([0,1])) \)-compatible for \( \alpha \in (\frac{1}{2},1) \).

**Proposition 2.1.** Let us consider a function \( f : [0, \infty) \to (0, \infty) \) that satisfies the following conditions:

\( (G_1) \) \( f \) is locally Lipschitz, and such that for all \( \sigma > 0 \), the following properties hold:

(i) \( f \in C^0([0, \sigma]) \), for some \( \theta \in (0,1) \) (independent of \( \sigma \)), and

(ii) \( \exists \omega(\sigma) > 0 \) such that
\[
f(y) - f(x) > -\omega(\sigma)(y-x), \quad 0 \leq x < y \leq \sigma.
\]

\( (G_2) \) \( x \mapsto f(x)/x \) is strictly decreasing on \((0, \infty)\).

\( (G_3) \) \( \lim_{x \to \infty} (f(x)/x) = b \), with \( b \geq 0 \).

Let us define \( a = \lim_{x \to 0} (f(x)/x) \), and \( \Lambda^* := (\frac{\Lambda_1}{a}, \frac{\Lambda_1}{b}) \).

If \( a < \infty \), then \( \mathfrak{F}_f = \{ \lambda f \}_{\lambda \in \Lambda^*} \) is \( (C^0_0,2\alpha([0,1]);C^2([0,1])) \)-compatible relatively to \( \Lambda^* \), for \( \alpha \in (\frac{1}{2},1) \).

**Proof.** This proposition is a direct consequence of the theory of sectorial operators and analytic semigroups [Lun95, Paz83, SY02, Tai95], and the theory of sublinear elliptic equations [BO86].

Consider \( \Lambda = [0, \infty) \), and \( f_{\lambda} = \lambda f \), for \( \lambda \in [0, \infty) \). Then from [Tai98, Theorem 5] which generalizes the “classical” result of [BO86, Theorem 1], we have that
\[
-\partial_{xx}^2 u = \lambda f(u), \quad u(0) = u(1) = 0,
\]
has a unique solution \( u \in C^2([0,1]) \) if and only if
\[
(2.13) \quad \frac{\lambda_1}{a} < \lambda < \frac{\lambda_1}{b},
\]
where \( \lambda_1 \) is the first eigenvalue of \(-\partial_{xx}^2\) with Dirichlet condition.

Let us consider \( \Lambda^* := (\frac{\Lambda_1}{a}, \frac{\Lambda_1}{b}) \). The realization of the Laplace operator \( A = -\partial_{xx}^2 \) in \( X = C([0,1]) \) with domain,
\[
(2.14) \quad D(A) = C^0_0,2\alpha([0,1]) := \{ u \in C^0,2\alpha([0,1]) : u(0) = u(1) = 0 \},
\]
is sectorial for \( \alpha \in (\frac{1}{2},1) \), and therefore generates an analytic semigroup on \( X \); see [Lun95].
The theory of analytic semigroups shows that under the aforementioned assumptions on $f$, for every $u_0 \in C_0^{0,2\alpha}([0,1])$, there exists a unique solution $u_\lambda \in C^1([0,\tau_\lambda(u_0)); C^2([0,1]))$ of $(P_{f_\lambda})$ defined on a maximal interval $[0,\tau_\lambda(u_0))$, with $\tau_\lambda(u_0) > 0$ (and $f_\lambda = \lambda f$); see e.g. [LLMP05, Proposition 6.3.8]. Since our assumptions on $f$ imply that there exists $C > 0$ such that $0 \leq f(x) \leq C(1 + x)$ for all $x \geq 0$, from [LLMP05, Proposition 6.3.5] we can deduce that $\tau_\lambda(u_0) = \infty$.

Let us introduce now,

\[
D(f_\lambda) := \{u_0 \in C_0^{0,2\alpha}([0,1]) : \sup_{t > 0} \|u_\lambda(t; u_0)\|_{C^2([0,1])} < \infty\},
\]

then $S_\lambda(t) : D(f_\lambda) \to D(f_\lambda)$, defined by $S_\lambda(t)u_0 = u_\lambda(t; u_0)$ is well defined for all $t \geq 0$, and for all $u_0 \in D(f_\lambda)$. Furthermore $\{S_\lambda(t)\}_{t \geq 0}$ is a (nonlinear) semigroup on $D(f_\lambda)$, in the sense that $S_\lambda \in C(D(f_\lambda), D(f_\lambda))$,

\[
\begin{align*}
S_\lambda(t + s) &= S_\lambda(t)S_\lambda(s), \ \forall \ t, s \geq 0, \\
S_\lambda(0) &= \operatorname{Id}_{C_0^{0,2\alpha}([0,1])},
\end{align*}
\]

and the map $t \mapsto S_\lambda(t)u_0$ belongs to $C([0,\infty), D(f_\lambda))$.

It is now easy to verify from what precedes that (ii) and (iii) of Definition 2.1 are satisfied.

We have thus proved that $\mathfrak{F}_f = \{\lambda f\}_{\lambda \in \Lambda^*}$ is $(C_0^{0,2\alpha}([0,1]); C^2([0,1]))$-compatible relatively to $\Lambda^*$, for $\alpha \in (\frac{1}{2}, 1)$.  

\(\Box\)

**Remark 2.3.** Let us remark that if furthermore $\lambda b > \lambda_1^{-1}$, it can be proved — based on Lyapunov functions techniques [CH98] and the non-increase of lap-number of solutions for scalar semilinear parabolic problems [Mat82] — that there exists at least one solution $u$ to $(P_{f_\lambda})$ emanating from some $u_0 \in C_0^{0,2\alpha}([0,1])$ for which $u$ does not remain in any bounded set for all time [Ben10, Lemma 10.1, Remark 10.2], and hence becomes unbounded in infinite time. It is the possible occurrence of such a phenomena that motivated to include a boundedness requirement in the definition of $D(f_\lambda)$ in (2.15).

**Example 2.2.** Let $g_\varepsilon(x) = \exp(x/(1 + \varepsilon x))$. A simple calculation shows that for $x \neq 0$,

\[
\left(\frac{g_\varepsilon(x)}{x}\right)' = -\frac{x^2}{x^2(1 + \varepsilon x)^2}\left(\frac{\varepsilon^2 x^2 + (2\varepsilon - 1)x + 1}{1 + \varepsilon x}\right),
\]

which implies in particular that $g_\varepsilon(x)/x$ is strictly decreasing for all $x > 0$ if $\varepsilon > 1/4$. Note also that condition $(G_1)$ of Proposition 2.1 holds, and that $b = 0$ and $a = \infty$ in this case.

Even if $a = \infty$, a (global) semigroup can still be defined (for each $\lambda \in (0, \infty)$) on the subset $D(\lambda g_\varepsilon)$ such as given in (2.15) with $f_\lambda = \lambda g_\varepsilon$. From the proof of Proposition 2.1, it is easy then to deduce that the family $\{\lambda g_\varepsilon\}_{\lambda \in (0, \infty)}$ is in fact $(C_0^{0,2\alpha}([0,1]); C^2([0,1]))$-compatible relatively to $(0, \infty)$ for $\alpha \in (\frac{1}{2}, 1)$ and for $\varepsilon > 1/4$.

Hereafter, $X$ and $Y$ will be two Banach spaces with respective norms denoted by $\| \cdot \|_X$ and $\| \cdot \|_Y$; and $\Omega$ will be an open bounded subset of $\mathbb{R}^d$, such that the following elliptic problem

\[
\begin{align*}
-\Delta u &= f_\lambda(u), \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

makes sense in $Y$. We introduce below a concept of topological equivalence between families of semilinear parabolic problems for $(X; Y)$-compatible families of nonlinearities.
Definition 2.2. Let $\Lambda$ be a metric space and $I$ be an unbounded interval of $\mathbb{R}$. Let $N(I, \mathbb{R})$ be a set of functions from $I$ to $\mathbb{R}$. Consider two families $\{f_\lambda\}_{\lambda \in \Lambda^*}$ and $\{\hat{f}_\lambda\}_{\lambda \in \hat{\Lambda}^*}$ of $N(I, \mathbb{R})$, which are both $(X; Y)$-compatible relatively to $\Lambda^*$ and $\hat{\Lambda}^*$ respectively.

For each $\lambda \in \Lambda^*$ and $\hat{\lambda} \in \hat{\Lambda}^*$, one denote by $\{S_\lambda(t)\}_{t \geq 0}$ and $\{\hat{S}_{\lambda}(t)\}_{t \geq 0}$, the semigroups acting on $D(f_\lambda)$ and $D(\hat{f}_\lambda)$, and by $(P_{f_\lambda})$ and $(P_{\hat{f}_\lambda})$. One denote finally by $\mathcal{S}_f$ and by $\mathcal{S}_{\hat{f}}$, the respective family of such semigroups.

Then $\mathcal{S}_f$ and $\mathcal{S}_{\hat{f}}$ are called topologically equivalent if there exists an homeomorphism $H : \Lambda \times \bigcup_{\lambda \in \Lambda^*} D(f_\lambda) \to \Lambda \times \bigcup_{\lambda \in \hat{\Lambda}^*} D(\hat{f}_\lambda)$, such that $H(\lambda, u) = (p(\lambda), H_\lambda(u))$ where $p$ and $H_\lambda$ satisfy the following two conditions:

(i) $p$ is an homeomorphism from $\Lambda^*$ to $\hat{\Lambda}^*$,

(ii) for all $\lambda \in \Lambda^*$, $H_\lambda$ is an homeomorphism from $D(f_\lambda)$ to $D(\hat{f}_{p(\lambda)})$, such that,

$$\forall \lambda \in \Lambda^*, \forall u_0 \in D(f_\lambda), \forall t > 0, H_\lambda(S_\lambda(t)u_0) = \hat{S}_{p(\lambda)}(t)H_\lambda(u_0).$$

(2.18)

In case of such equivalence, the families of problems $\{(P_{f_\lambda})\}_{\lambda \in \Lambda^*}$ and $\{(P_{\hat{f}_\lambda})\}_{\lambda \in \hat{\Lambda}^*}$ will be also referred as topologically equivalent.

Remark 2.4. Note that the relation of topological equivalence given by (2.18) may be relaxed as follows,

$$\forall \lambda \in \Lambda, \forall u_0 \in D(f_\lambda), H_\lambda(S_\lambda(t)u_0) = \hat{S}_{p(\lambda)}(\gamma(t, u_0))H_\lambda(u_0),$$

(2.19)

where $\gamma : [0, \infty) \times D(f_\lambda) \to [0, \infty)$ is an increasing and continuous function of the first variable.

This second approach is an extension of the concept of topological orbital equivalence, classically encountered in finite-dimensional dynamical systems theory, which allows, in particular, for systems presenting periodic orbits of different periods, to be equivalent; avoiding by this way the so-called problem of modulii; see [KH97].

To the opposite, the topological equivalence relation (2.18) excludes this possibility, which can be viewed as too restrictive for general semigroups, at a first glance. However, for semigroups generated by semilinear parabolic equations over open bounded domain, due to their gradient-like structure [CH98, Hal88, Rob01], this problem of modulii does not occur since the $\omega$-limit set of each semigroup is typically included into the set of its equilibria [CH98, Hal88, Rob01].

Definition 2.3. Let $\mathcal{S}_f$ be a family of semigroups as defined in Definition 2.2. Let $\mathcal{E}_f$ be the corresponding family of equilibria, in the sense that,

$$\mathcal{E}_f := \{ (\lambda, \phi_\lambda) \in \Lambda \times D(f_\lambda) : S_\lambda(t)\phi_\lambda = \phi_\lambda, \forall t \in (0, \infty) \}.$$  

(2.20)

Assume that $\Lambda$ is an unbounded interval of $\mathbb{R}$. A fold-point on $\mathcal{E}_f$ is a point $(\lambda^*, u^*) \in \mathcal{E}_f$, such that there exists a local continuous map $\mu : s \in (-\varepsilon, \varepsilon) \mapsto (\lambda(s), u(s))$ for some $\varepsilon > 0$, verifying the following properties:

(F1) For all $s \in (-\varepsilon, \varepsilon)$, one has $(\lambda(s), u(s)) \in \mathcal{E}_f$, with $(\lambda(0), u(0)) = (\lambda^*, u^*)$.

(F2) $s \mapsto \lambda(s)$ has a unique extremum on $(-\varepsilon, \varepsilon)$ attained at $s = 0$.

(F3) There exists $r^* > 0$ such that for all $0 < r < r^*$, the set

$$\partial \mathcal{B}((\lambda^*, u^*); r) \cap \{ \mu(s), s \in (-\varepsilon, \varepsilon) \},$$
has cardinal two; where

$$\mathcal{B}((\lambda^*, u^*); r) := \{ (\lambda, u) \in \mathbb{R} \times D(f_{\lambda}), \ : \ |\lambda - \lambda^*| + \|u - u^*\|_X < r \}. \quad (2.21)$$

**Definition 2.4.** Let $\mathcal{G}_f$ be a family of semigroups as defined in Definition 2.2. Let $\mathcal{E}_f$ be the corresponding family of equilibria given by (2.20). Assume that $\Lambda$ is an unbounded interval of $\mathbb{R}$. Let $n$ be an integer such that $n \geq 3$. A multiple-point with $n$ branches on $\mathcal{E}_f$ is a point $(\lambda^*, u^*) \in \mathcal{E}_f$, such that there exists at most $n$ local continuous maps

$$\mu_i : s \in (-\varepsilon_i, \varepsilon_i) \mapsto (\lambda_i(s), u_i(s)) \text{ for some } \varepsilon_i > 0, \ i \in \{1, \ldots, n\},$$

verifying the following properties:

(G1) $\mu_i \neq \mu_j$ for all $i \neq j$.

(G2) For all $i \in \{1, \ldots, n\}$, and for all $s \in (-\varepsilon_i, \varepsilon_i)$, one has $(\lambda_i(s), u_i(s)) \in \mathcal{E}_f$, with $(\lambda_i(0), u_i(0)) = (\lambda^*, u^*)$.

(G3) There exists $r^* > 0$ such that for all $0 < r < r^*$, the set

$$\partial \mathcal{B}((\lambda^*, u^*); r) \cap \bigcup_{i \in \{1, \ldots, n\}} \{ \mu_i(s), \ s \in (-\varepsilon_i, \varepsilon_i) \},$$

has cardinal $n$, where $\mathcal{B}((\lambda^*, u^*); r)$ is as given in (2.21).

**Remark 2.5.** The terminologies of Definition 2.3 and Definition 2.4 will be also adopted for one-parameter families of steady states in $Y$ as introduced in Definition 2.1 (ii).

Based on these definitions, simple criteria of non topological equivalence between two families of semigroups can be then formulated. The proposition below whose proof is left to the reader’s discretion, summarizes these criteria.

**Proposition 2.2.** Assume $\Lambda$ is an unbounded interval of $\mathbb{R}$. Let $\mathcal{G}_f$ and $\mathcal{G}_f^*$ be two families of semigroups as defined in Definition 2.2. Let $\mathcal{E}_f$ and $\mathcal{E}_f^*$ be the corresponding families of equilibria. Then $\mathcal{G}_f$ and $\mathcal{G}_f^*$ are not topologically equivalent if one of the following conditions are fulfilled.

(i) $\mathcal{E}_f$ is constituted by a single unbounded continuum in $\Lambda \times X$, and $\mathcal{E}_f^*$ is the union of at least two disjoint unbounded continua in $\Lambda \times X$.

(ii) $\mathcal{E}_f$ and $\mathcal{E}_f^*$ are each constituted by a single continuum, and the set of fold-points of $\mathcal{E}_f$ and $\mathcal{E}_f^*$ are not in one-to-one correspondence.

(iii) $\mathcal{E}_f$ and $\mathcal{E}_f^*$ are each constituted by a single continuum, and there exists an integer $n \geq 3$ such that the set of multiple-points with $n$ branches of $\mathcal{E}_f$ and $\mathcal{E}_f^*$ are not in one-to-one correspondence.

We can now introduce the following notion of *topological robustness* to small perturbations, and the related notion of *topological instability* of families of semigroups generated by one-parameter families of semilinear parabolic problems.

**Definition 2.5.** Let $\Lambda$ be a metric space and $I$ be an unbounded interval of $\mathbb{R}$. Let $\mathcal{N}(I, \mathbb{R})$ be a set of functions from the interval $I$ to $\mathbb{R}$ endowed with a topology $\mathcal{T}$. Consider a family $\mathcal{F}_f = \{ f_{\lambda} \}_{\lambda \in \Lambda^*}$ of $\mathcal{N}(I, \mathbb{R})$ which is $(X; Y)$-compatible relatively to $\Lambda^* \subset \Lambda$.

For each $\lambda \in \Lambda^*$, one denote by $\{ S_{\lambda}(t) \}_{t \geq 0}$ the semigroup acting on $D(f_{\lambda})$, and by $\mathcal{G}_f$ the corresponding family of semigroups, and generated respectively by $(P_{f_{\lambda}})$ and $(P_{f_{\lambda}}^*)$.

We will say that $\mathcal{F}_f$ is $\mathcal{T}$-stable with respect to perturbations in the $\mathcal{T}$-topology, if for each $\lambda \in \Lambda^*$, there exists in this topology a neighborhood $\mathcal{U}_\lambda$ of $f_{\lambda}$ such that for any neighborhood $\mathcal{U}_\lambda \subset \mathcal{U}_\lambda^*$,
Given a TR-connected and open subset of $(X;Y)$, Proposition 3.1.

In case where $\mathfrak{S}_f$ is $\mathfrak{T}$-stable, we will say furthermore that $\mathfrak{S}_f$ is $\mathfrak{T}$-topologically robust in $X$, with respect to perturbations in the $\mathfrak{T}$-topology, if

(ii) for any family $\mathfrak{S}_f = \{\tilde{f}_\lambda\}_{\lambda \in \Lambda^*}$ such that for all $\lambda \in \Lambda^*$, $\tilde{f}_\lambda \in \mathfrak{U}_\lambda$, the families of semigroups, $\mathfrak{S}_\tilde{f}$ and $\mathfrak{S}_f$, are topologically equivalent in the sense of Definition 2.2.

Given a $\mathfrak{T}$-stable family $\mathfrak{S}_f$, if for any $\mathfrak{U}_\lambda$ neighborhood of $f_\lambda$ such as provided by (i), there exists $\tilde{f}_\lambda \in \mathfrak{U}_\lambda$ for which $\mathfrak{S}_\tilde{f}$ and $\mathfrak{S}_f$ are not topologically equivalent, then $\mathfrak{S}_f$ will be called topologically unstable with respect to small perturbations in $\mathfrak{T}$.

3. Topologically unstable families of semilinear parabolic problems: Main result

Similar conjectures such as recalled in Section 2.1 regarding the qualitative change of a “cusp-type” for the $\lambda$-bifurcation diagram have been pointed out in other semilinear elliptic problems (see e.g. [BCT88b, ZWS07]), but general conditions on the nonlinear term under which a given family of semilinear parabolic problems is topologically unstable with respect to small perturbations remain to be clarified; see however [Dan08].

It is the purpose of Theorem 3.2 below to identify such conditions. As already mentioned its proof relies on a combination of Theorem A.1 proved in Appendix A of this article, the growth property of the branch of minimal solutions as recalled in Proposition 3.1 below, and methods of generation of a discontinuity in the minimal branch borrowed from the proof of [CEP02, Theorem 1.2]. Theorem 3.2 allows us to conclude to the existence of a broad class of topologically unstable families of semilinear parabolic problems, not necessarily related to a specific type of bifurcation, and for situations where a global attractor is not guaranteed to exist. Figure 1 below depicts some typical $\lambda$-bifurcation diagrams of the corresponding families of semilinear elliptic problems concerned with Theorem 3.2.

It is worth mentioning that the proof of Theorem 3.2 provides furthermore the mechanism at the origin of the aforementioned topological instability. This mechanism boils down essentially to a local deformation of the $\lambda$-bifurcation diagram — associated with (3.1) below — by the creation of either a multiple-point or a new fold-point, when an appropriate small perturbation is applied on the nonlinear term $g$. This is accomplished under assumptions on $g$ that prevents the use of linearization techniques; see Remark 3.2 below.

To prepare the proof of Theorem 3.2, we first recall classical results about the solution set of,

\[
\begin{cases}
-\Delta u = \lambda g(u), & \text{in } \Omega, \lambda \geq 0, \\
u|_{\partial \Omega} = 0,
\end{cases}
\]

(3.1)

summarized into the Proposition 3.1 below. The proof of this proposition, based on the use of sub- and super-solutions methods, can be found in [Caz06, Theorem 3.4.1].

**Proposition 3.1.** Consider a locally Lipschitz function $g : [0, \infty) \to (0, \infty)$. Let $\Omega$ be a bounded, connected and open subset of $\mathbb{R}^d$. Then there exists $0 < \lambda^* \leq \infty$ with the following properties.

(i) For every $\lambda \in [0, \lambda^*)$, there exists a unique minimal solution $u_\lambda \geq 0$, $u_\lambda \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (3.1). The solution $u_\lambda$ is minimal in the sense that any supersolution $v \geq 0$ of (3.1) satisfies $v \geq u_\lambda$.

(ii) The map $\lambda \mapsto u_\lambda$ is increasing from $(0, \infty)$ to $H_0^1(\Omega) \cap L^\infty(\Omega)$.
(iii) If \( \lambda^* < \infty \) and \( \lambda > \lambda^* \), then there is no solution of (3.1) in \( H^1_0(\Omega) \cap L^\infty(\Omega) \).

If \( \Omega \) is furthermore connected, then \( \lambda^* = \infty \) if \( \lim_{u \to \infty} \frac{g(u)}{u} = 0 \), and \( \lambda^* < \infty \) if \( \lim_{u \to \infty} \inf g(u) > 0 \).

**Remark 3.1.** [Caz06, Theorem 3.4.1] is in fact proved for functions \( g \) which are \( C^1 \). From this proof, it is not difficult to see however that the conclusions of this proposition still hold by assuming \( f \) to be locally Lipschitz instead of \( C^1 \).

We are now in position to prove our main theorem.

**Theorem 3.2.** Consider a continuous, locally Lipschitz, and increasing function \( g : [0, \infty) \to (0, \infty) \). Let \( \Omega \) be a bounded and open domain of \( \mathbb{R}^d \), with \( d = 1 \) or \( d = 2 \). Let \( \Lambda = [0, \infty) \) and let \( \Lambda^* = [0, \lambda^*) \) with \( \lambda^* \) be as defined by Proposition 3.1. Assume that the solution set

\[
E_g := \{ (\lambda, \phi) \in [0, \lambda^*) \times C^2(\overline{\Omega}) : -\Delta \phi = \lambda g(\phi), \phi|_{\partial \Omega} = 0, \phi > 0 \text{ in } \Omega \},
\]

is well defined for some \( \alpha \in (0, 1) \) and is constituted by a continuum without multiple-points on it.

Assume furthermore that the set of fold-points of \( E_g \) given by

\[
F := \{ (\lambda, u_\lambda) : (\lambda, u_\lambda) \text{ is a fold-point of } E_g \},
\]

satisfies one of the following conditions

(i) \( F \neq \emptyset \), \( 0 < \lambda_m := \min\{ \lambda \in (0, \lambda^*) : F_\lambda \neq \emptyset \} < \lambda^* \), and

\[
E_g \cap \Gamma^-_{\lambda_m} = \text{minimal branch of } E_g,
\]

where

\[
\Gamma^-_{\lambda_m} = \{ (\lambda, \phi) \in (0, \infty) \times C^2(\overline{\Omega}) : \lambda < \lambda_m, \|\phi\|_\infty < \|u_{\lambda_m}\|_\infty \}.
\]

(ii) \( F \neq \emptyset \) and there exists \( \lambda^*_2 \in (0, \lambda^*) \) for which there exists \( \{ (\lambda, u_\lambda) \}_{\lambda \in (\lambda^*_2, \lambda^*)} \subset E_g \) such that \( \lim_{\lambda \to \lambda^*_2} \|u_\lambda\|_\infty = \infty \), with \( E_g \cap \Gamma^-_{\lambda^*_2} = \text{minimal branch of } E_g \).

(iii) \( F = \emptyset \) and \( E_g \) is constituted only by its minimal branch.
Finally, assume that the family of functions \( \tilde{\mathcal{F}}_g := \{ \lambda g \}_{\lambda \in [0, \lambda^*)} \) is \((X; C^{2,\alpha}(\overline{\Omega}))\)-compatible relatively to \([0, \lambda^*)\) for some Banach space \(X\), and that this family is \(\mathcal{T}\)-stable in the sense of Definition 2.5, with \(\mathcal{T}\) denoting the \(C^0\)-compact-open topology on \(C(\mathbb{R}^+, \mathbb{R}^+)\).

Let \(\mathcal{E}_g\) be the corresponding family of semigroups \(\{ S_\lambda(t) \}_{\lambda \in [0, \lambda^*)}\) associated with

\[
\begin{align*}
\partial_t u - \Delta u &= \lambda g(u), & \text{in } \Omega, \\
 u &= 0, & \text{on } \partial \Omega.
\end{align*}
\]

(3.5)

Then \(\mathcal{E}_g\) is topologically unstable with respect to small perturbations\(^{10}\) in \(\mathcal{T}\). Furthermore, such a perturbation \(\tilde{g}\) can be chosen such that \(\tilde{g}\) is \(C^1\) and increasing, \(g - \tilde{g}\) is with compact support, and for which \(\mathcal{E}_{\tilde{g}}\) contains a new fold-point or a new multiple-point compared with \(\mathcal{E}_g\), for either \(\lambda \in (0, \lambda_m)\), or \(\lambda \in (0, \lambda_2)\), or \(\lambda \in (0, \lambda^*)\), depending on whether case (i), case (ii), or case (iii), is respectively concerned.

**Proof.** Let \(\mathcal{E}_g\) be the solution set in \([0, \lambda^*) \times C^{2,\alpha}(\overline{\Omega})\) of (3.1), i.e.,

\[
\mathcal{E}_g = \{ (\lambda, u_\lambda) \in [0, \lambda^*) \times C^{2,\alpha}(\overline{\Omega}) : -\Delta u_\lambda = \lambda g(u_\lambda), u_\lambda > 0 \text{ in } \Omega, \ u_\lambda|_{\partial \Omega} = 0, \}.
\]

First, note that by assumptions on \(\tilde{\mathcal{F}}_g\), we have for each \(\lambda \in [0, \lambda^*)\) the existence of \(D(\lambda g) \subset X\) such that Eq. (3.5) generates a semigroup acting on \(D(\lambda g)\); see Definition 2.1. By introducing \(D(\lambda g) = D(\lambda g) \cap \{ \phi > 0 \text{ in } \Omega \}\), we can still define a semigroup \(\{ S_\lambda(t) \}_{t \geq 0}\) acting on \(\tilde{D}(\lambda g)\), due to the maximum principle.

Let us recall now that [CEP02, Theorem 1.2] ensures, in dimension \(d = 1\) or \(d = 2\), the existence for all \(\varepsilon > 0\) of a \(C^1\), positive and increasing \(\varepsilon\)-perturbations \(\tilde{g}\) of \(g\) in the \(C^0\)-compact-open topology, for which the branch of minimal positive solutions \(\lambda \in (0, \lambda^*) \mapsto \tilde{u}_\lambda\),

\[
\begin{align*}
-\Delta u &= \lambda g(u), & u > 0 \text{ in } \Omega, \\
 u|_{\partial \Omega} &= 0,
\end{align*}
\]

(3.6)

undergoes a discontinuity of first kind, as a map from \((0, \tilde{\lambda}^*)\) to \(C^{2,\alpha}(\overline{\Omega})\).\(^{11}\)

More precisely, let \(\lambda_\varepsilon\) be chosen in \((0, \lambda^*)\). Given \(\varepsilon > 0\), [CEP02, Theorem 1.2] ensures the existence of an increasing \(C^1\) positive function \(\tilde{g}\), such that the following conditions hold:

\[
\begin{align*}
\| g - \tilde{g} \|_\infty &\leq \varepsilon, \\
\text{supp}(g - \tilde{g}) &\subset [\| u_{\lambda_\varepsilon} \|_\infty, \| u_{\lambda_\varepsilon} \|_\infty + \varepsilon],
\end{align*}
\]

for which the following set

\[
\mathcal{M} = \{ \tilde{u}_\lambda, \lambda \in [\tilde{\lambda}^*, \lambda_\varepsilon) \}.
\]

is constituted by minimal solutions of (3.6) over an interval \([\tilde{\lambda}^*, \lambda_\varepsilon)\) such that

\[
\tilde{\lambda}^* > \lambda_\varepsilon, \quad \tilde{u}_\lambda = u_{\lambda_\varepsilon} \text{ for } \lambda \in (0, \lambda_\varepsilon), \text{ and } \lambda \mapsto \tilde{u}_\lambda \text{ is discontinuous on } [\lambda_\varepsilon, \lambda_\varepsilon + \varepsilon].
\]

Conditions \((H_1)-(H_2)\) indicate that the perturbation \(\tilde{g}(x)\) of \(g(x)\) is localized for \(x\)-values located near \(\| u_{\lambda_\varepsilon} \|_\infty\) for some \(\lambda_\varepsilon\), and Condition \((H_3)\) shows that such a perturbation generates a discontinuity near \(\lambda_\varepsilon\), on the minimal branch associated with (3.6).

**Case (i).** We consider

\[
\mathcal{F} = \{ (\lambda, u_\lambda) : (\lambda, u_\lambda) \text{ is a fold-point of } \mathcal{E}_g \},
\]

and assume first that \(\mathcal{F} \neq \emptyset\) and that the condition (i) such as formulated in the statement of the theorem, is satisfied.

\(^{10}\)In the sense of Definition 2.5.

\(^{11}\)In [CEP02] the authors have proved the existence of such a discontinuity in the \(L^\infty(\Omega)\)-norm for solutions considered in \(C^2(\overline{\Omega})\) which is therefore valid for solutions considered in \(C^{2,\alpha}(\overline{\Omega})\).
Let us choose $\varepsilon > 0$ and $\lambda_s$ such that,

$$0 < \lambda_s + 2\varepsilon \leq \lambda_m := \min\{\lambda : (\lambda, u_\lambda) \in F\}$$

(3.7) and such that

$$\|u_{\lambda_k}\|_\infty + \varepsilon < \|u_{\lambda_m}\|_\infty.$$  

(3.8)

The latter is possible by monotony of the minimal branch; see Proposition 3.1.

For this choice of $\lambda_s$ and $\varepsilon$, and for the corresponding perturbation $\tilde{g}$ of $g$ verifying Conditions (H₁)-(H₃), it can be proved that there exists an unbounded continuum in $\hat{\Lambda}^s \times V$ of nontrivial solutions of (3.6) which emanates from $(0, 0\nu)$, with here $V = C^{2,\alpha}(\overline{\Omega})$, by application of Theorem A.1 and by using similar arguments as provided for the perturbed Gelfand problem (2.1) in Section 2.1.

Let $\lambda_c \in [\lambda_s, \lambda_s + \varepsilon]$ be the critical parameter value at which the discontinuity of the minimal branch, $\lambda \mapsto \hat{u}_\lambda$, takes place. Let $\hat{C}$ be the unbounded continuum of $\mathcal{E}_{\tilde{g}}$ which contains $(0, 0\nu)$. By construction of $\tilde{g}$ and assumption on $\mathcal{E}_{\tilde{g}}$, we deduce that

$$\hat{C} \cap \Gamma_{\lambda_s}^- = \{(\lambda, \hat{u}_\lambda)\}_{\lambda < \lambda_s},$$

(3.9)

where $\Gamma_{\lambda_s}^-$ is defined as in Eq. (3.4), by replacing $\lambda_m$ with $\lambda_s$. Hereafter, we define the set $\Gamma_{\lambda_c}^-$ similarly.

Assume first that,

$$\{(\lambda, \hat{u}_\lambda)\}_{\lambda < \lambda_c} \not\subseteq \hat{C} \cap \Gamma_{\lambda_c}^-,$$

then necessarily there exists a branch of solutions of (3.6) that intercepts the set

$$\{(\lambda, \hat{u}_\lambda)\}_{\lambda < \lambda < \lambda_c},$$

at some point $(\lambda, \hat{u}_\lambda)$ for $\lambda \in [\lambda_s, \lambda_c)$, leading to the existence of a multiple-point of $\mathcal{E}_{\tilde{g}}$ which turns out to be a signature of topological instability of $\mathcal{E}_{\tilde{g}}$ according to Proposition 2.2-(iii) and the assumption made on $\mathcal{E}_{\tilde{g}}$.

Consider now the case where

$$\{(\lambda, \hat{u}_\lambda)\}_{\lambda < \lambda_c} = \hat{C} \cap \Gamma_{\lambda_c}^-.$$  

(3.10)

A more careful analysis is here required to conclude to the topological instability of $\mathcal{E}_{\tilde{g}}$.

First, let us note that standard compactness arguments allow us to conclude to the existence of a sequence $\{\lambda_k\}$, such that

$$v_{\lambda_c} := \lim_{\lambda_k \uparrow \lambda_c} \hat{u}_{\lambda_k} \text{ exists},$$

and such that this limit is a solution of (3.6) for $\lambda = \lambda_c$.

This solution has to be the minimal solution at $\lambda_c$ since by construction of [CEP02],

$$\lim_{\lambda \uparrow \lambda_c} \|\hat{u}_\lambda\|_\infty < \lim_{\lambda \downarrow \lambda_c} \|\hat{u}_\lambda\|_\infty.$$  

(3.11)

Therefore,

$$v_{\lambda_c} = \hat{u}_{\lambda_c} \text{ and } (\lambda_c, \hat{u}_{\lambda_c}) \in \hat{C}.$$  

(3.12)

Denote by $A^+_{\lambda_c}$ the point $(\lambda_c, \lim_{\lambda \uparrow \lambda_c} \hat{u}_{\lambda})$ which exists from same arguments of compactness.

Similarly we get that $A^+_{\lambda_c} = (\lambda_c, \hat{u}^+_{\lambda_c})$ for some $\hat{u}^+_{\lambda_c} \in \mathcal{E}_{\tilde{g}}$.

Since $\hat{u}^+_{\lambda_c} = \lim_{\lambda \downarrow \lambda_c} \hat{u}_{\lambda_c}$ and $\lambda_c < \lambda_m$ by construction, and since the map $\lambda \mapsto \hat{u}_\lambda$ is increasing from Proposition 3.1-(ii), we infer that necessarily,

$$\|\hat{u}^+_{\lambda_c}\|_\infty \leq \|\hat{u}_{\lambda_m}\|_\infty.$$  

(3.13)
In other words, the right-hand limit at the critical parameter value \( \lambda_c \) of the minimal solutions of the perturbed problem (3.6), comes with less energy than the energy of the first\(^{12}\) fold-point associated with the unperturbed problem (3.1).

Since \( \hat{C} \) is unbounded in \( \Lambda \times V \), either \( (\lambda_c, \hat{u}_{\lambda_c}) \) is a fold-point of \( \hat{C} \) that lives thus according to (3.13) in \( \Gamma_{\lambda_m}^\lambda \), or \( (\lambda_c, \hat{u}_{\lambda_c}) \) is not a fold-point of \( \hat{C} \) and \( \hat{C} \cap \Gamma_{\lambda,\gamma}^+ \neq \emptyset \) for all \( \gamma > 0 \), where

\[
\Gamma_{\lambda,\gamma}^+: = \left\{ (\lambda, v) \in \Lambda \times V : \lambda > \lambda_c, \| v - \hat{u}_{\lambda_c} \| < \gamma \right\}.
\]

Let us show that the second option of this alternative does not hold. By contradiction, assume that \( \hat{C} \cap \Gamma_{\lambda,\gamma}^+ \neq \emptyset \) for all \( \gamma > 0 \) and that \( (\lambda_c, \hat{u}_{\lambda_c}) \) is not a fold-point of \( \hat{C} \), then condition (F\(_2\)) of Definition 2.3 is violated and therefore any local continuous map given for some \( \theta > 0 \) as,

\[
\mu : s \in (-\theta, \theta) \mapsto (\lambda(s), v(s)),
\]

and such that for all \( s \in (-\theta, \theta) \), \( (\lambda(s), v(s)) \in \hat{C} \) with \( (\lambda(0), v(0)) = (\lambda_c, \hat{u}_{\lambda_c}) \), comes with its underlying map

\[
s \mapsto \lambda(s),
\]

that does not attain its maximum at \( s = 0 \).

Recall from Eq. (3.11) that

\[
\| \hat{u}_{\lambda_c} \|_\infty < \| \hat{u}_{\lambda_c}^+ \|_\infty.
\]

Then by continuity of the map \( \mu \) there exists \( 0 < \beta \leq \theta \) such that \( s \mapsto \lambda(s) \) is strictly increasing on \((0, \beta)\) and such that

\[
\| v(s) \|_\infty < \| \hat{u}_{\lambda_c}^+ \|_\infty, \quad \forall s \in (0, \beta).
\]

This last inequality is in contradiction with the minimality property of the branch \( \lambda \mapsto \hat{u}_\lambda \) and the fact that \( \| \hat{u}_\lambda \|_\infty \geq \| \hat{u}_{\lambda_c}^+ \|_\infty \) for any \( \lambda > \lambda_c \) such that \( \lambda - \lambda_c \) is small enough, by construction of \( \hat{u}_{\lambda_c}^+ \).

Thus, the second part of the aforementioned alternative does not hold which implies that \( (\lambda_c, \hat{u}_{\lambda_c}) \) is a fold-point of \( \hat{C} \) that lives according to (3.13) in \( \Gamma_{\lambda_m}^\lambda \). By definition of \( \lambda_m \) in (3.7), no fold-point exists in \( \Gamma_{\lambda_m}^\lambda \) for \( \mathcal{E}_g \). On the other hand, recall that by construction of \( \hat{g} \) satisfying (H\(_1\))-(H\(_3\)) for \( \varepsilon \) and \( \lambda_\varepsilon \) satisfying (3.7)-(3.8), one has that \( g(x) = \hat{g}(x) \) for \( x > \| \hat{u}_{\lambda_m} \|_\infty \) and hence

\[
\mathcal{E}_{\hat{g}} \cap \Gamma_{\lambda_m}^+ = \mathcal{E}_g \cap \Gamma_{\lambda_m}^+,
\]

where

\[
\Gamma_{\lambda_m}^+ := \left\{ (\lambda, \phi) \in (0, \infty) \times C^{2,\alpha}(\Omega) : \lambda > \lambda_m, \| \phi \|_\infty > \| \hat{u}_{\lambda_m} \|_\infty \right\}.
\]

As a consequence, the set of fold-points in \( \Gamma_{\lambda_m}^+ \) of \( \mathcal{E}_{\hat{g}} \) and \( \mathcal{E}_g \) are identical. We have just proved the existence of a fold-point of \( \mathcal{E}_{\hat{g}} \) in \((0, \lambda_m) \times X\) which no longer exists — in a homeomorphic sense — on \( \mathcal{E}_g \) by definition of \( \lambda_m \). From Proposition 2.2-(i), we conclude that \( \mathcal{E}_g \) and \( \mathcal{E}_{\hat{g}} \) are thus not topologically equivalent.

**Case (ii).** The proof follows the same lines than above by working with \((0, \lambda_\varepsilon)\) instead of \((0, \lambda_m)\), and by localizing the perturbation on \( \hat{C} \cap \Gamma_{\lambda_\varepsilon}^- \).

**Case (iii).** If \( \mathcal{F} = \emptyset \), \( \lambda_\varepsilon \) may be chosen arbitrary in \((0, \lambda^*)\), and we can proceed as preceding to create a fold-point of \( \mathcal{E}_{\hat{g}} \) whereas \( \mathcal{E}_g \) does not possess any fold-point (\( \mathcal{F} = \emptyset \)).

In all the cases, we are thus able to exhibit a perturbation \( \hat{g} \) such that \( \| g - \hat{g} \|_\infty \leq \varepsilon \) and \( \mathcal{E}_g \) and \( \mathcal{E}_{\hat{g}} \) are not topologically equivalent, for any choice of \( \varepsilon > 0 \). We have thus proved that \( \mathcal{E}_{\hat{g}} \) is topologically unstable in the sense of Definition 2.5. The proof is complete.\(^\Box\)

\(^{12}\)As \( \lambda \) is increased from 0.
Remark 3.2. If we assume \( g \) to be \( C^1 \) instead of continuous and locally Lipschitz, it can be shown that necessarily \( (\lambda_c, \widehat{u}_{\lambda_c}) \) obtained in the proof above, is degenerate in the sense that
\[
\lambda_1(-\Delta - \lambda_c g'(\widehat{u}_{\lambda_c})I) = 0,
\]
and the linearized equation has a nontrivial solution. Then under further assumptions on \( g \) and appropriate a priori bounds, the existence of a fold-point at \( (\lambda_c, \widehat{u}_{\lambda_c}) \) can be ensured using e.g. [CR75, Theorem 1.1]; see also [CR73, OS99].

The regularity assumption on \( g \) of Theorem 3.2 prevents the use of such linearization techniques. Note that parabolic problems with locally Lipschitz nonlinearity are commonly encountered in energy balance models; see [RCCS14] and references therein.

Theorem A.1 serves here as a substitutive ingredient to cope with the lack of regularity caused by our assumptions on \( g \). At the same time, it is unclear how to weaken further these assumptions, since the proof of Theorem 3.2 provided above has made a substantial use of the growth property of the minimal branch such as recalled in Proposition 3.1 above; see also Remark 3.1.

The possibility of creation of a discontinuity in the minimal branch by arbitrarily small perturbations of the nonlinearity, has played a crucial role in the proof of Theorem 3.2. This is made possible when the spatial dimension is equal to one or two, due to the following observation regarding a specific Poisson equation used in the creation of a discontinuity in the minimal branch such as proposed in [CEP02].

Given \( r > 0 \), we denote by \( B_r \) the ball of \( \mathbb{R}^d \) of radius \( r \), centered at the origin. For \( 0 < \rho < R \), the solution \( \Psi_\rho \) of the following Poisson equation
\[
\begin{cases}
-\Delta \Psi_\rho = 1_{B_\rho}, & \text{in } B_R, \\
\Psi_\rho|_{\partial B_R} = 0,
\end{cases}
\]
satisfies for \( \rho < R/2 \),
\[
\inf_{B_{2\rho}} \Psi_\rho = \rho^2 K(\rho),
\]
where the behavior of \( K(\rho) \) as \( \rho \to 0 \) is of the form
\[
K(\rho) \approx \begin{cases} R/\rho, & \text{if } d = 1, \\
|\log \rho|/2, & \text{if } d = 2.
\end{cases}
\]
This can be proved by simply writing down the analytic expression of the solution to (3.18); see [CEP02, Lemma 3.1]. When \( d \geq 3 \), \( K(\rho) \) converges to a constant (depending on \( d \)) as \( \rho \to 0 \). This removal of the singularity at 0 for \( K \), is responsible of the “sufficiently large” requirement regarding the perturbation of the nonlinearity, in order to achieve a discontinuity in the minimal branch by the techniques of [CEP02] in dimension \( d \geq 3 \). Whether this point is purely technical or more substantial, is still an open problem.

Appendix A. Unbounded continuum of solutions to parametrized fixed point problems, in Banach spaces

We communicate in this appendix on a general result concerning the existence of an unbounded continuum of fixed points associated with one-parameter families of completely continuous perturbations of the identity map in a Banach space. This theorem is rooted in the seminal work of [LS34] that initiated what is known today as the Leray-Schauder continuation theorem. Extensions of such a continuation result can be found in [FMP86, MP84] for the multi-parameter case. Theorem A.1 below, summarizes such a result in the one-parameter case whose proof is given here for the sake of completeness. Under a nonzero condition on the Leray-Schauder degree to hold at some parameter value, Theorem A.1 ensures in particular the existence of
an unbounded continuum of solutions to nonlinear problems for which the nonlinearity is not necessarily Fréchet differentiable.

Results similar to Theorem A.1 below, regarding the existence of an unbounded continuum of solutions to nonlinear eigenvalue problems, have been obtained in the literature, see e.g. [Rab71, Theorem 3.2], [Rab74, Corollary 1.34], [BB80, Theorem 3] or [Ama76, Theorem 17.1]. Similar to these works, the ingredients for proving Theorem A.1 rely also on the Leray-Schauder degree properties and connectivity arguments from point set topology. However, following [FMP86, MP84], Theorem A.1 ensures the existence of an unbounded continuum of solutions to parameterized fixed point problems under more general conditions on the nonlinear term than required in these works.

Hereafter, deg(Ψ, O, y) stands for the classical Leray-Schauder degree of Ψ with respect to O and y which is well defined for completely continuous perturbations Ψ of the identity map of a Banach space E, if y ∈ Ψ(∂O), when O is an open bounded subset of E; see e.g. [Dei85, Nir01]. In what follows the λ-section of a nonempty subset A of R⁺×E, will be defined as:

(A.1) \[ A_λ := \{ u ∈ E : (λ, u) ∈ A \}. \]

**Theorem A.1.** Let \( U \) be an open bounded subset of a real Banach space \( E \) and assume that \( G : R⁺×E \to E \) is completely continuous (i.e. compact and continuous). We assume that there exists \( λ₀ ≥ 0 \), such that the equation,

(A.2) \[ u - G(λ₀, u) = 0 \]

has a unique solution \( u₀ \), and,

(A.3) \[ \deg(I - G(λ₀, ·), U, 0) \neq 0. \]

Let us introduce

(A.4) \[ S^+ = \{ (λ, u) ∈ [λ₀, ∞) × E : u = G(λ, u) \}. \]

Then there exists a continuum \( C^+ ⊆ S^+ \) (i.e. a closed and connected subset of \( S^+ \)) such that the following properties hold:

(i) \( C^+_0 \cap U = \{ u₀ \} \),

(ii) Either \( C^+ \) is unbounded or \( C^+_0 \cap (E \setminus \overline{U}) \neq ∅ \).

In order to prove this theorem, we need an extension of the standard homotopy property of the Leray-Schauder degree [Dei85, Nir01] to homotopy cylinders that exhibit variable λ-sections. This is the purpose of the following Lemma.

**Lemma A.1.** Let \( O \) be a bounded open subset of \([λ₁, λ₂] × E\), and let \( G : \overline{O} \to E \) be a completely continuous mapping. Assume that \( u \neq G(λ, u) \) on \( ∂O \), then for all \( λ \in [λ₁, λ₂] \),

\[ \deg(I - G(λ, ·), O_λ, 0_E) \]

is independent of \( λ \), where \( O_λ = \{ u ∈ E : (λ, u) ∈ O \} \) is the \( λ \)-section of \( O \).

**Proof.** We may assume, without loss of generality, that \( O \neq ∅ \) and that \( λ₁ = \inf \{ λ : O_λ \neq ∅ \} \)

and \( λ₂ = \sup \{ λ : O_λ \neq ∅ \} \). Consider \( ε > 0 \) and the following superset of \( O \) in \( R × E \),

(A.5) \[ O^ε := O \cup (\{ λ₁ - ε, λ₁ \} × O₁ λ₁ \cup (λ₂, λ₂ + ε) × O₂ λ₂). \]

Then \( O^ε \) is an open bounded subset of \( R × E \). Since \( O \) is closed by definition and \( G \) is continuous, then according to the Dugundgi extension theorem on metric spaces [Dug66, Thm. 6.1 p. 188] (cf. Lemma B.2 below), \( G \) can be extended to \( R × E \) as a continuous function that we denote by \( \overline{G} \).
Now consider,  
\[ \forall (\lambda, u) \in \mathbb{R} \times E, \quad H(\lambda, u) := (\lambda - \lambda^*; u - \tilde{G}(\lambda, u)), \]

with some arbitrary fixed \( \lambda^* \in [\lambda_1, \lambda_2] \). Then \( H \) is a completely continuous perturbation of the identity\(^{13}\) in \( \mathbb{R} \times E \). We denote by \( \bar{E} \) the set \( \mathbb{R} \times E \) in what follows.

Since \( H(\lambda, u) = 0_{\bar{E}} \) if and only if \( \lambda = \lambda^* \) and \( u = \tilde{G}(\lambda, u) \), and since \( \lambda^* \in [\lambda_1, \lambda_2] \) and \( G(\lambda, u) \neq u \) on \( \partial O \) by assumptions, we can conclude that,

\[ (A.6) \quad \forall (\lambda, u) \in \partial O^\varepsilon, \quad H(\lambda, u) \neq 0_{\bar{E}}. \]

Therefore \( \text{deg}(H, O^\varepsilon, 0_{\bar{E}}) \) is well defined and constant.

Let us consider the following one-parameter family \( \{H_t\}_{t \in [0, 1]} \) of perturbations of \( H \) defined by,

\[ \forall (\lambda, u) \in \mathbb{R} \times E, \quad H_t(\lambda, u) := (\lambda - \lambda^*; u - t\tilde{G}(\lambda, u) - (1 - t)\tilde{G}(\lambda^*, u)). \]

Then

\[ (A.7) \quad H_t(\lambda, u) = 0 \iff (\lambda = \lambda^* \text{ and } u = \tilde{G}(\lambda^*, u)), \]

and from our assumptions, we conclude again that \( H_t(\lambda, u) \neq 0_{\bar{E}} \) for all \( (\lambda, u) \in \partial O^\varepsilon \) and all \( t \in [0, 1] \).

By applying now the standard homotopy invariance principle [Dei85, Nir01] to the family \( \{H_t\}_{t \in [0, 1]} \) we have

\[ (A.8) \quad \text{deg}(H_1, O^\varepsilon, 0_{\bar{E}}) = \text{deg}(H, O^\varepsilon, 0_{\bar{E}}) = \text{deg}(H_0, O^\varepsilon, 0_{\bar{E}}). \]

Let \( K \) be the closed subset of \( \bar{O}^\varepsilon \) such that \( O^\varepsilon \setminus K = (\lambda_1 - \varepsilon, \lambda_2 + \varepsilon) \times O_{\lambda^*} \). Then \( 0_{\bar{E}} \) does not belong to \( H(\partial O^\varepsilon \cup K) \) since the cancelation of \( H \) is possible only on the \( \lambda^* \)-cross section, while \( K \) does not intercept this section by construction and \( 0_{\bar{E}} \notin H(\partial O^\varepsilon) \) from (A.6). By applying now the excision property of the Leray-Schauder degree [Dei85, Nir01] with such a \( K \), we obtain,

\[ (A.9) \quad \text{deg}(H_0, O^\varepsilon, 0_{\bar{E}}) = \text{deg}(H_0, (\lambda_1 - \varepsilon, \lambda_2 + \varepsilon) \times O_{\lambda^*}, 0_{\bar{E}}). \]

The interest of (A.9) relies on the fact that the degree is by this way expressed on a cartesian product which allows us to apply the cartesian product formula (see Lemma B.1), which gives in our case

\[ (A.10) \quad \text{deg}(H_0, (\lambda_1 - \varepsilon, \lambda_2 + \varepsilon) \times O_{\lambda^*}, 0_{\bar{E}}) = \text{deg}(I - G(\lambda^*, \cdot), O_{\lambda^*}, 0_E), \]

since \( \text{deg}(f, (\lambda_1 - \varepsilon, \lambda_2 + \varepsilon), 0_E) = 1 \) with \( f(\lambda) = \lambda - \lambda^* \), and \( \lambda^* \in [\lambda_1, \lambda_2] \).

By applying now (A.10), (A.9) and (A.8) and by recalling that \( \text{deg}(H, O^\varepsilon, 0_{\bar{E}}) \) is independent of \( \lambda^* \), we have thus proved that for arbitrary \( \lambda^* \in [\lambda_1, \lambda_2] \), \( \text{deg}(I - G(\lambda^*, \cdot), O_{\lambda^*}, 0_E) \) is also independent of \( \lambda^* \). The proof is complete. \( \square \)

**Remark A.1.** The introduction of \( O^\varepsilon \) such as defined in (A.5) above was used in order to work within an open bounded subset of a Banach space, here \( \mathbb{R} \times E \), and thus to work within the framework of the Leray-Schauder degree\(^{14}\). The Dugundgi theorem is used to appropriately extend the mapping \( G \) to \( O^\varepsilon \) in order to apply the Leray-Schauder degree techniques.

\(^{13}\)This statement can be proved by relying on the construction of the continuous extension used in the proof of the Dugundgi theorem. For the sake of completeness, we sketch the proof of the latter in Appendix B; see Lemma B.2.

\(^{14}\)the original open subset \( O \) is not an open subset of a Banach space, but of the metric space \( [\lambda_1, \lambda_2] \times E \).
The last ingredient to prove Theorem A.1, is the following separation lemma from point set topology (Lemma A.2 below). A separation of a topological space $X$ is a pair of nonempty open subsets $U$ and $V$, such that $U \cap V = \emptyset$ and $U \cup V = X$. A space is connected if it does not admit a separation. Two subsets $A$ and $B$ are connected in $X$ if the exists a connected set $Y \subset X$, such that $A \cap Y \neq \emptyset$ and $B \cap Y \neq \emptyset$. Two nonempty subsets $A$ and $B$ of $X$ are separated if there exists a separation $U, V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$. There exists a relationship between these concepts in the case where $X$ compact, this is summarized in the following separation lemma.

**Lemma A.2.** (Separation lemma) If $X$ is compact and $A$ and $B$ are not separated, then $A$ and $B$ are connected in $X$.

The proof of this lemma may be found in [Dei85, Lemma 29.1]; see also [Kur68].

As a result if two subsets of a compact set are not connected, they are separated. We are now in position to prove Theorem A.1.

**Proof of Theorem A.1.**

**Proof.** Let $\mathcal{C}^+$ be the maximal connected subset of $\mathcal{S}^+$ such that (i) holds, which is trivial by assumptions. We proceed by contradiction. Assume that $\mathcal{C}^+_{\lambda_0} \cap (E \setminus \mathcal{U}) = \emptyset$ and that $\mathcal{C}^+$ is bounded in $[\lambda_0, \infty) \times E$. Then there exists a constant $R > 0$ such that for each $(\lambda, u) \in \mathcal{C}^+$ we have $\|u\| + |\lambda| < R$. Introduce,

$$S^+_{2R} := \{(\lambda, u) \in \mathcal{S}^+ : \|u\| + |\lambda| \leq 2R\}.$$ 

From the complete continuity of $G$ it follows that any set of the form $\mathcal{H} := \{(\lambda, u) \in \Lambda \times E : u = G(\lambda, u)\}$, with $\Lambda$ a closed and bounded subset of $[\lambda_0, \infty)$, is a compact subset of $[\lambda_0, \infty) \times E$. As a result, $S^+_{2R}$ is a compact subset of $[\lambda_0, \infty) \times E$.

There are two possibilities. Either (a) $S^+_{2R} = \mathcal{C}^+$ or, (b) there exists $(\lambda^*, u^*) \in S^+_{2R}$ such that $(\lambda^*, u^*)$ does not belong to $\mathcal{C}^+$.

Let $\mathcal{U}$ be as defined in Theorem A.1. Consider case (b) first. We want to apply Lemma A.2 with $X = S^+_{2R}$, $A = \mathcal{C}^+$, and $B = \{\lambda^*\} \times S^+_{2R}$. Obviously, $A$ and $B$ are not connected in $S^+_{2R}$ since $(\lambda^*, u^*) \notin \mathcal{C}^+$ and $\mathcal{C}^+$ is the maximal connected subset of $S^+$. We may therefore apply Lemma A.2 in such a case and build an open subset $\mathcal{O}$ of $[\lambda_0, \infty) \times E$, such that the following properties hold,

- (c1) $\mathcal{O}_{\lambda_0} = \mathcal{U}$ (since $\mathcal{C}^+_{\lambda_0} \cap (E \setminus \mathcal{U}) = \emptyset$),
- (c2) $\mathcal{C}^+ \subset \mathcal{O}$,
- (c3) $S^+_{2R} \cap \partial \mathcal{O} = \emptyset$ and,
- (c4) $\mathcal{O}_{\lambda^*}$ contains no solutions of $u = G(\lambda^*, u)$.

The last property comes from the fact that $A$ and $B$, as defined above, are separated.

From (c3), we get by applying Lemma A.1, that,

$$\forall \lambda \in \Lambda_R, \quad \text{deg}(I - G(\lambda, \cdot), \mathcal{O}_\lambda, 0) = \text{deg}(I - G(\lambda_0, \cdot), \mathcal{O}_{\lambda_0}, 0),$$

where $\Lambda_R$ denotes the projection of $S^+_{2R}$ onto $[\lambda_0, \infty)$.

Now $\text{deg}(I - G(\lambda_0, \cdot), \mathcal{O}_{\lambda_0}, 0) \neq 0$ by (c1) and the assumptions of Theorem A.1. We obtain therefore a contradiction from (c4) when (A.11) is applied for $\lambda = \lambda^*$.

The case $\mathcal{C}^+ = S^+_{2R}$ may be treated along the same lines and is left to the reader. The proof is complete. $\square$

**Remark A.2.** Theorem A.1 shows in particular that if for all $\mathcal{U}$ there is a unique solution $(\lambda_0, u_0)$ in $\mathcal{U}$, of $u = G(\lambda_0, u)$, then there exists an unbounded continuum of solutions of $u = G(\lambda, u)$, provided that there exists an open set $\mathcal{V}$ in $E$ such that $\text{deg}(I - G(\lambda_0, \cdot), \mathcal{V}, 0) \neq 0$. 

Remark A.3. It is not essential that \( u_0 \) be the only solution of (A.2) in \( U \). If one only assumes (A.3), one obtains a similar conclusion about the existence of possibly finitely many continua satisfying the alternative formulated in (ii) of Theorem A.1.

**Appendix B. Product formula for the Leray-Schauder degree, and the Dugundji extension theorem**

This appendix contains auxiliaries lemmas used in the previous Appendix. We first start with the cartesian product formula for the Leray-Schauder degree.

**Lemma B.1.** Assume that \( U = U_1 \times U_2 \) is a bounded open subset of \( E_1 \times E_2 \), where \( E_1 \) and \( E_2 \) are two Banach spaces with \( U_1 \) and \( U_2 \) open subsets of \( E_1 \) and \( E_2 \) respectively. Suppose that for all \( x = (x_1, x_2) \in E \), \( f(x) = (f_1(x_1), f_2(x_2)) \), where \( f_1 : \overline{U_1} \to E_1 \) and \( f_2 : \overline{U_2} \to E_2 \) are continuous and suppose that \( y = (y_1, y_2) \in E \) is such that \( y_1 \) (resp. \( y_2 \)) does not belong to \( f_1(\partial U_1) \) (resp. \( f_2(\partial U_2) \)). Then,

\[
\deg(f, U, y) = \deg(f_1, U_1, y_1) \deg(f_2, U_2, y_2).
\]

We recall below the Dugundgi extension theorem [Dug66, Thm. 6.1 p. 188].

**Lemma B.2. (Dugundgi)** Let \( E \) and \( X \) be Banach spaces and let \( f : \mathcal{C} \to E \) a continuous mapping, where \( \mathcal{C} \) is a closed subset of \( E \). Then there exists a continuous mapping \( \bar{f} : E \to K \) such that \( \bar{f}(u) = f(u) \) for all \( u \in \mathcal{C} \).

**Proof.** (Sketch) For each \( u \in E \setminus \mathcal{C} \), let \( r_u = \frac{1}{2}\text{dist}(u, \mathcal{C}) \), and \( B_u := \{ v \in E : \|v - u\| < r_u \} \). Then \( \text{diam}(B_u) \leq \text{dist}(B_u, \mathcal{C}) \), and \( \{ B_u \}_{u \in E \setminus \mathcal{C}} \) is a open cover of \( E \setminus \mathcal{C} \) which admits a local refinement \( \{ \mathcal{O}_\lambda \}_{\lambda \in \Lambda} \); i.e. \( \bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda \supset E \setminus \mathcal{C} \). For each \( \lambda \in \Lambda \) there exists \( B_u \) such that \( B_u \supset \mathcal{O}_\lambda \), and every \( u \in E \setminus \mathcal{C} \) has a neighborhood \( U \) such that \( U \) intersects at most finitely many elements of \( \{ \mathcal{O}_\lambda \}_{\lambda \in \Lambda} \) (locally finite family).

Introduce now \( \gamma : E \setminus \mathcal{C} \to [0, \infty) \), defined by \( \gamma(u) = \sum_{\lambda \in \Lambda} \text{dist}(u, \overline{\mathcal{O}_\lambda}) \) and introduce

\[
\forall \lambda \in \Lambda, \forall u \in E \setminus \mathcal{C}, \quad \gamma_\lambda(u) = \frac{\text{dist}(u, \overline{\mathcal{O}_\lambda})}{\gamma(u)}.
\]

By construction, the above sum over \( \Lambda \) contains only finitely many terms and thus \( \gamma \) is continuous.

Now define \( \bar{f} \) by,

\[
\bar{f} = \begin{cases} 
  f(u), & \text{if } u \in \mathcal{C}, \\
  \sum_{\lambda \in \Lambda} \gamma_\lambda(u) f(u_\lambda), & u \notin \mathcal{C}.
\end{cases}
\]

Then it can be shown that \( \bar{f} \) is continuous.

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