Identification and reconstruction of oscillatory modes in U.S. business cycles using Multivariate Singular Spectrum Analysis

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in cooperation with
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Motivation

Given

- Short observations
- with different scales in space and time
- with observation errors
- many simultaneous measurements

Questions

- Distinguish between regular deterministic behavior “cycles” and irregular behavior “noise”
- Extract a skeleton of the underlying system
- Understand and model the underlying mechanism
Motivation

U.S. macroeconomic data from the Bureau of Economic Analysis — Trend residuals after Hodrick-Prescott detrending
**Motivation**

Origin of business cycles?

**Exogenous**
Exogenous shocks cause fluctuations in a stable system
- Real business cycle theory

**Endogenous**
Intrinsic dynamics leads to instability and to oscillatory behavior
- Endogenous business cycle theory

U.S. macroeconomic data from the Bureau of Economic Analysis — Trend residuals after Hodrick-Prescott detrending

Oscillatory modes in U.S. business cycles
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Outline

1 **Singular Spectrum Analysis**
   - Reconstruction of dynamical behavior
   - Spectrum of eigenvalues
   - Reconstruction of oscillatory modes
   - Properties and interpretation

2 **Multivariate Singular Spectrum Analysis**
   - Extraction of shared behavior
   - Multichannel spectral analysis
   - Reconstruct skeleton of dynamical system
   - Recurrence analysis

3 **Conclusions**
Outline

1. **Singular Spectrum Analysis**
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2. **Multivariate Singular Spectrum Analysis**
   - Extraction of shared behavior
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   - Recurrence analysis

3. **Conclusions**
Singular Spectrum Analysis
Reconstruction of dynamical behavior

Idea

- Singular Spectrum Analysis relies on classical Karhunen-Loève spectral decomposition of a stochastic process
  (Karhunen 1946, Loève 1945)
- Broomhead and King introduced M-SSA into dynamical systems analysis
  (Broomhead and King 1986a,b)
- Robust version of Mañé-Takens idea to reconstruct dynamics via time-delayed embedding
  (Mañé 1981, Takens 1981)

Observation: Time series \( \{ x(n) \} \), of length \( n = 1 \ldots N \)
Singular Spectrum Analysis
Reconstruction of dynamical behavior

**Idea**

- Singular Spectrum Analysis relies on classical Karhunen-Loève spectral decomposition of a stochastic process
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- Robust version of Mañé-Takens idea to reconstruct dynamics via time-delayed embedding
  \[ \text{(Mañé 1981, Takens 1981)} \]

**Observation:** Time series \( \{ x(n) \} \), of length \( n = 1 \ldots N \)

**Embedding:** \( M \)-dimensional time-delayed embedding

\[
X(n) = [x(n), x(n + 1), \ldots, x(n + M - 1)]
\]
How to extract information from $X$?
Singular Spectrum Analysis
Reconstruction of dynamical behavior

How to extract information from $X$?

**Singular Spectrum Analysis (SSA)**

1. Compute covariance matrix $C = X^T X / N$
2. Eigendecomposition $\Lambda = E^T C E$
   - diagonal matrix $\Lambda$ with eigenvalues $\lambda_k$
   - orthogonal matrix $E$ with eigenvectors $e_k$ in columns
3. Projection of $X$ onto $E$ gives principal components (PCs)
4. Reconstruct dynamical behavior in $x$ that belongs to $e_k$ with reconstructed components (RCs)

(Vautard, Yiou, Ghil 1992; Plaut & Vautard 1994)
Singular Spectrum Analysis
Spectrum of eigenvalues

- Eigenvalues quantify the variance into the direction of corresponding eigenvector
- Important dynamical behavior is in largest eigenvalues/-vectors

Idea: Separate “signal” from “noise”

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Singular Spectrum Analysis
Reconstruction of oscillatory modes

- Eigenvalues 1–4 explain already 50% of total variance
- Reconstruct time series with these eigenvectors
Singular Spectrum Analysis
Reconstruction of oscillatory modes

- Stepwise reconstruction of the dynamical behavior
- Reconstruct ghost limit cycles of dynamical systems (albeit unstable, the orbit is visited by system’s trajectory)

(Kimoto and Ghil 1993; Ghil and Yiou 1996)
Singular Spectrum Analysis
Eigenvectors

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### Properties and Interpretation

#### SSA

- **Analyzing function**
  - Basis function: Eigenvectors $e_k$
  - Decomposition: $\sum_{m=1}^{M} x(n + m - 1)e_k(m)$
  - Scale: $n$
  - Epoch: $m$

- **Wavelet analysis**
  - Mother wavelet: $\Psi$
  - $\Psi$ chosen a priori
  - Decomposition: $\int x(t)\Psi(t - b/a)dt$
  - Scale: $a$
  - Epoch: $b$

#### (after Ghil et al. 2002)
Singular Spectrum Analysis
Properties and interpretation

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<th>Wavelet analysis</th>
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<td>Analyzing function</td>
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(after Ghil et al. 2002)

- The basis of decomposition is not chosen a priori
- Eigenvectors originate from the covariance matrix $C$
- Eigenvectors can form pairs of even/uneven functions, analog to sine/cosine-pairs in Fourier analysis
- Eigenvectors are frequency-selective filters, which adapt to important oscillatory modes of the time series
Singular Spectrum Analysis
Properties and interpretation

- Eigenvectors are data-adaptive, frequency-selective filters
- Plot eigenvalue vs. dominant frequency of corresponding eigenvector
- Estimation of power spectrum
Multivariate Singular Spectrum Analysis

1. Singular Spectrum Analysis
   Reconstruction of dynamical behavior
   Spectrum of eigenvalues
   Reconstruction of oscillatory modes
   Properties and interpretation

2. Multivariate Singular Spectrum Analysis
   Extraction of shared behavior
   Multichannel spectral analysis
   Reconstruct skeleton of dynamical system
   Recurrence analysis

3. Conclusions

Oscillatory modes in U.S. business cycles

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Motivation

U.S. macroeconomic data from the Bureau of Economic Analysis — Trend residuals after Hodrick-Prescott detrending
Multivariate SSA is a natural extension of SSA

(Broomhead and King 1986b)

It extracts principal oscillations in time and space

**Observation:** Multivariate time series \( \{x_d(n)\} \)

\[ d = 1 \ldots D \text{ channels of length } n = 1 \ldots N \]
Multivariate Singular Spectrum Analysis
Extraction of shared behavior

- Multivariate SSA is a natural extension of SSA
  (Broomhead and King 1986b)

- It extracts principal oscillations in time and space

**Observation:** Multivariate time series \( \{x_d(n)\} \)

\[ d = 1 \ldots D \text{ channels of length } n = 1 \ldots N \]

**Embedding:** \( M \)-dimensional time-delayed embedding of each channel

\[ X_d(n) = (x_d(n), x_d(n + 1), \ldots, x_d(n + M - 1)) \]

Full augmented trajectory matrix

\[ X = \begin{pmatrix} X_1 & X_2 & \ldots & X_D \end{pmatrix} \]
Multivariate Singular Spectrum Analysis (M-SSA)

1. Compute covariance matrix $C = \frac{X^TX}{N}$
2. Eigendecomposition $\Lambda = E^TCE$
   - diagonal matrix $\Lambda$ with eigenvalues $\lambda_k$
   - orthogonal matrix $E$ with eigenvectors $e_k$ in columns
3. Projection of $X$ onto $E$ gives principal components (PCs)
4. Reconstruct dynamical behavior in $x$ that belongs to $e_k$ with reconstructed components (RCs)

- Same eigendecomposition of $C$ as before for SSA
- Except that $C$ includes also cross correlations
  - Eigenvectors are multivariate filters
  - M-SSA detects common spectral properties

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Multivariate Singular Spectrum Analysis
Extraction of shared behavior

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Multivariate Singular Spectrum Analysis
Multichannel spectral analysis

- Find common spectral components ▶ Similar dynamics
- Detection of synchronization clusters

(Groth & Ghil, 2011)
Multivariate Singular Spectrum Analysis
Reconstruct skeleton of dynamical system

Oscillatory modes in U.S. business cycles

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• Similar behavior of the economy in the history
• Recurrence analysis in phase space
Conclusions

Advantages

- M-SSA helps to extract common oscillatory modes in a large set of observations
- Data-adaptive method with no a-priori basis functions
- Reconstruct “skeleton” of dynamics in complete multivariate fashion
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- M-SSA helps to extract common oscillatory modes in a large set of observations
- Data-adaptive method with no a-priori basis functions
- Reconstruct “skeleton” of dynamics in complete multivariate fashion

Verification of results

- Check robustness of results with respect to embedding parameter $M$
- Apply statistical significance tests: Monte Carlo SSA
- Do not consider M-SSA as stand-alone method ⇒ Reexamine with other methods (e.g. wavelet analysis) and their tests
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