Multivariate singular spectrum analysis and the road to phase synchronization

Andreas Groth\(^{(1)}\)
and Michael Ghil\(^{(1,2,3)}\)

(1) Laboratoire de Meteorologie Dynamique, Ecole Normale Superieure, Paris, France
(2) Geosciences Department, Ecole Normale Superieure, Paris, France
(3) Department of Atmospheric & Oceanic Sciences and Institute of Geophysics & Planetary Physics, University of California, Los Angeles, USA

EGU General Assembly 2010
Motivation

Given

- Short data sets
- Different scales in space and time
- Observation errors

Wanted

- Enhance signal-to-noise ratio
- Distinguish between regular deterministic behavior ("cycles") and irregular behavior ("noise")
- Extract skeleton of the underlying system(s)
  - Derive conclusions about synchronization mechanism
Motivation

U.S. macroeconomic data from the Bureau of Economic Analysis — Trend residuals after Hodrick-Prescott detrending
Motivation of singular spectrum analysis (SSA)

Problem

- Underlying dynamical systems $\dot{\mathbf{y}} = F(\mathbf{y})$ is unknown
- We obtain a scalar measurement $x(t)$ with $t = 1 \ldots N$
  from this system $x(t) = H(y(t))$
Motivation of singular spectrum analysis (SSA)

Problem
- Underlying dynamical systems $\dot{y} = F(y)$ is unknown
- We obtain a scalar measurement $x(t)$ with $t = 1 \ldots N$ from this system $x(t) = H(y(t))$

Idea
- Build a new $M$-dimensional time series from $x(t)$ and its lagged copies

$$x(t) = (x(t), x(t + 1), \ldots, x(t + M - 1))$$

- These delay coordinates $x$ share key topological properties with $y$ (Whitney 1936, Mañé 1981, Takens 1981 ...)

Motivation M-SSA Phase synchronization Conclusions
Motivation of singular spectrum analysis (SSA)

**Problem**
- Underlying dynamical systems $\dot{y} = F(y)$ is unknown
- We obtain a scalar measurement $x(t)$ with $t = 1 \ldots N$ from this system $x(t) = H(y(t))$

**Idea**
- Build a new $M$-dimensional time series from $x(t)$ and its lagged copies

$$\mathbf{x}(t) = (x(t), x(t + 1), \ldots, x(t + M - 1))$$

- These delay coordinates $\mathbf{x}$ share key topological properties with $y$ \cite{Whitney1936, Mane1981, Takens1981}

**Question**
- How to extract properties from short and noisy time series?
Singular spectrum analysis

Idea Apply PCA to $x(t)$ in order to extract the entire attractor of the (nonlinear) system

(Broomhead and King, 1986a,b)
Singular spectrum analysis

Idea
Apply PCA to $x(t)$ in order to extract the entire attractor of the (nonlinear) system

(Broomhead and King, 1986a,b)

However
An original motivation to interpret the results in terms of attractor dimensions fails even in relatively simple cases!

(Vautard and Ghil, 1989)
Singular spectrum analysis

**Idea**  Apply PCA to $x(t)$ in order to extract the entire attractor of the (nonlinear) system  

(Broomhead and King, 1986a,b)

**However**  An original motivation to interpret the results in terms of attractor dimensions fails even in relatively simple cases!  

(Vautard and Ghil, 1989)

**Anyway**  The idea to reconstruct the skeleton of the attractor, e.g. the most robust, albeit unstable limit cycles by means of PCA remains promising  

(Vautard and Ghil 1989, Ghil and Vautard 1991)
Delay embedding of

\[ x(t) \rightarrow \mathbf{x}(t) = (x(t), x(t + 1), \ldots, x(t + M - 1)) \in \mathbb{R}^M \]
Singular spectrum analysis

1. **Delay embedding of**

\[ x(t) \rightarrow \mathbf{x}(t) = (x(t), x(t + 1), \ldots, x(t + M - 1)) \in \mathbb{R}^M \]

2. **Estimate covariance matrix** \( \mathbf{C} \) of \( \mathbf{x} \) (of size \( M \times M \)) with elements

\[ c_{ij} = \frac{1}{N - |i - j|} \sum_{t=1}^{N-|i-j|} x(t)x(t + |i - j|) \]
Singular spectrum analysis

1. Delay embedding of

\[ x(t) \rightarrow \mathbf{x}(t) = (x(t), x(t + 1), \ldots, x(t + M - 1)) \in \mathbb{R}^M \]

2. Estimate covariance matrix \( \mathbf{C} \) of \( \mathbf{x} \) (of size \( M \times M \)) with elements

\[
c_{ij} = \frac{1}{N - |i - j|} \sum_{t=1}^{N-|i-j|} x(t) x(t + |i - j|)
\]

3. Determine eigenvalues and eigenvectors \( (\lambda_k, \mathbf{e}_k) \) of \( \mathbf{C} \)

\[
\mathbf{C}\mathbf{e}_k = \lambda_k \mathbf{e}_k \quad k = 1 \ldots M
\]
Delay embedding of

\[ x(t) \rightarrow \mathbf{x}(t) = (x(t), x(t + 1), \ldots, x(t + M - 1)) \in \mathbb{R}^M \]

Estimate **covariance matrix** \( \mathbf{C} \) of \( \mathbf{x} \) (of size \( M \times M \)) with elements

\[ c_{ij} = \frac{1}{N - |i - j|} \sum_{t=1}^{N-|i-j|} x(t)x(t + |i - j|) \]

Determine **eigenvalues and eigenvectors** \((\lambda_k, \mathbf{e}_k)\) of \( \mathbf{C} \)

\[ \mathbf{C}\mathbf{e}_k = \lambda_k \mathbf{e}_k \quad k = 1 \ldots M \]

\( \lambda_k \) equals the partial variance of \( \mathbf{x} \) into the direction of \( \mathbf{e}_k \)
Delay embedding of

\[ x(t) \rightarrow \mathbf{x}(t) = (x(t), x(t + 1), \ldots, x(t + M - 1)) \in \mathbb{R}^M \]

2. Estimate covariance matrix \( \mathbf{C} \) of \( \mathbf{x} \) (of size \( M \times M \)) with elements

\[ c_{ij} = \frac{1}{N - |i-j|} \sum_{t=1}^{N-|i-j|} x(t)x(t + |i-j|) \]

3. Determine eigenvalues and eigenvectors \( (\lambda_k, \mathbf{e}_k) \) of \( \mathbf{C} \)

\[ \mathbf{C}\mathbf{e}_k = \lambda_k \mathbf{e}_k \quad k = 1 \ldots M \]

- \( \lambda_k \) equals the partial variance of \( \mathbf{x} \) into the direction of \( \mathbf{e}_k \)
- Sum of all eigenvalues equals the total variance of the original time series \( x \)

M-SSA and phase synchronization
Andreas Groth, LMD, ENS, Paris
Spectrum of eigenvalues

- Eigenvalues $\lambda_k$ describe “importance” of phase space directions $e_k$
- Relevant dynamics is captured by largest eigenvalues
- Separate signal from noise
- Oscillatory pairs describe possible limit cycles
Reconstruct oscillatory behavior

- Reconstruct limit cycles of dynamical systems from oscillatory pairs of eigenvalues

(Kimoto and Ghil, 1993; Ghil and Yiou, 1996)
Phase synchronization

Synchronization of self-sustained (chaotic) oscillators

- Synchronization means that rhythms of detuned oscillators adjust
- Reflected in spectrum of Lyapunov exponents — but in practice not accessible
- In practice: Study of phases and frequencies
- Locking of mean frequencies and phases \( \Rightarrow \) Phase synchronization
Phase synchronization

**Synchronization of self-sustained (chaotic) oscillators**

- Synchronization means that rhythms of detuned oscillators adjust
- Reflected in spectrum of Lyapunov exponents — but in practice not accessible
  - In practice: Study of phases and frequencies
- Locking of mean frequencies and phases $\Rightarrow$ Phase synchronization

**M-SSA**

- Multivariate SSA extracts oscillatory behavior in phase space of multivariate time series
- These oscillations appear in M-SSA as oscillatory pairs of eigenvalues
  - Road to phase synchronization is well reflected in the spectrum of eigenvalues
Motivation

M-SSA

Phase synchronization

Conclusions

Two Rössler systems

\[ \lambda_1 + \lambda_2 = (\lambda_3 + \lambda_4) \]

M-SSA:

\(x, y, z\)-coordinates of two systems with \(M = 30\) give

180 eigenvalues

180 eigenvectors

M-SSA and phase synchronization

Andreas Groth, LMD, ENS, Paris
Two Rössler systems
Closed loop of five Rössler systems

M-SSA eigenvalues

\[
\lambda_k
\]

Frequencies of oscillators

\[
f
\]

Coupling strength

M-SSA and phase synchronization

Andreas Groth, LMD, ENS, Paris
Advantages of M-SSA in practical applications

- Helps to extract regular deterministic behavior such as limit cycles
- Is data-adaptive and needs no a priori basis functions (e.g. wavelets)
- No pre-analysis necessary in order to introduce an appropriate phase
- Deals very well with short and noisy time series
- No pre-filtering (such as denoising, band-pass filtering etc.) necessary as for phase by Hilbert transform
Conclusions

Advantages of M-SSA in practical applications

- Helps to extract regular deterministic behavior such as limit cycles
- Is data-adaptive and needs no a priori basis functions (e.g. wavelets)
- No pre-analysis necessary in order to introduce an appropriate phase
- Deals very well with short and noisy time series
- No pre-filtering (such as denoising, band-pass filtering etc.) necessary as for phase by Hilbert transform

Verification of results

- Check robustness of results by changing embedding dimension $M$
- Apply statistical significance tests: Monte Carlo M-SSA
- Do not consider M-SSA as stand-alone method $\Rightarrow$ Apply additional methods and their tests
References


M-SSA eigenvalues

\[ \lambda_k \]

Frequencies of oscillators

f

Coupling strength

M-SSA and phase synchronization

Andreas Groth, LMD, ENS, Paris