Persistence of hyperbolic fixed points for diffeomorphisms under composition

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Abstract

We reconsider here the persistence of hyperbolic fixed points for diffeomorphisms on $\mathbb{R}^n$ by treating their perturbations in a finer topology than the Fréchet topology of $\mathcal{C}^1$-convergence on compact subsets. More precisely, we construct a semi-pseudo metric of Čech, which induces such a topology on a subgroup of $\mathcal{C}^r$-diffeomorphisms, endowed with the law of composition. We show that, in this topology, hyperbolic fixed points persist within the given subgroup.

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Résumé

Sur la persistance de points fixes hyperboliques de difféomorphismes vis à vis de certaines perturbations sur $(\mathbf{Diff}^r(\mathbb{R}^n), \circ)$. La persistance des points fixes hyperboliques de difféomorphimes de $\mathbb{R}^n$ est montrée via un théorème des fonctions implicites, pour des perturbations mesurées à l'aide d'une topologie qui exploite la structure naturelle de groupe pour la composition, de l'ensemble des difféomorphismes. Plus précisément, nous introduisons une semi-pseudo métrique de Čech qui permet, dans un même temps, de définir des sous-groupes de difféomorphismes. La topologie induite sur ces derniers est plus fine que la topologie de Fréchet de la convergence $\mathcal{C}^1$ sur tout compact de $\mathbb{R}^n$, sans exiger d'ajout d'ordre de dérivation.

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1. Introduction

In a recent paper [2], we were led to consider the $\mathcal{C}^r$-conjugacy between time-T maps for the orbits of a T-periodic system of ordinary differential equations (ODEs) and those of an averaged form of this system. The conjugacy conditions formulated as Lemma 6.1 of [2] was a cornerstone for the rigorous justification of the nonlinear averaging procedure introduced in that paper.

Related to this conjugacy of time-T maps, we were also interested in the existence of a hyperbolic T-periodic limit cycle for the periodic ODE system, living in a neighborhood of a hyperbolic fixed point of the averaged system. Classical existence results of this type rely on local perturbations of the averaged vector field in the $\mathcal{C}^1$-topology and are linked to persistence of hyperbolic fixed points of diffeomorphisms; the latter are obtained for local perturbations by applying the implicit function theorem [3].

To deal, however, with non-local perturbations of diffeomorphisms on $\mathbb{R}^n$, such persistence results require the use of the Fréchet topology of $\mathcal{C}^1$-convergence on compact subsets of $\mathbb{R}^n$ (the Fréchet $\mathcal{C}^1$-topology for short). Moreover, perturbations used in numerical calculations that do exhibit persistence are not always compatible with the $\mathcal{C}^1$-topology.

In the present paper, we take a step toward better agreement between the numerical observations of [2] and analytically rigorous results on the persistence of hyperbolic fixed points. To do so, we construct on $\mathrm{Diff}^r(\mathbb{R}^n)$, the set of $\mathcal{C}^r$-diffeomorphisms on $\mathbb{R}^n$, a topology that is finer than the Fréchet $\mathcal{C}^1$-topology. This finer topology is constructed without adding further derivatives, and it allows us to obtain results on the persistence of hyperbolic fixed points for perturbations that satisfy this topology. The extension of these results from diffeomorphisms to vector fields and the rigorous justification of the averaging procedure of [2] will appear in forthcoming work.

We propose to overcome the difficulties inherent to the Fréchet framework by working with the composition law for non-local perturbations. To achieve this, we introduce an appropriate topology on $(\mathrm{Diff}^r(\mathbb{R}^n), \circ)$, the group of $\mathcal{C}^r$-diffeomorphisms on $\mathbb{R}^n$ under composition. The purpose of this Note is to do so and to prove therewith a theorem on the persistence of hyperbolic fixed points for a suitable class of diffeomorphisms in this framework. The present approach allows us to deal with global perturbations of diffeomorphisms on the whole of $\mathbb{R}^n$, while obtaining persistence results that are more consistent with a form of robustness to "small" perturbations in model parameters that we might expect on physical and numerical grounds (cf. *Remark* 4).

2. A semi-pseudo metric of Čech and the associated topology

For each integer $r \geq 0$, we denote by $\mathrm{Diff}^r(\mathbb{R}^n)$ the group of diffeomorphisms of class $\mathcal{C}^r$ on $\mathbb{R}^n$, with the convention that $\mathrm{Diff}^0(\mathbb{R}^n)$ stands for the set of homeomorphisms of $\mathbb{R}^n$; we will also refer to homeomorphisms as $\mathcal{C}^0$-diffeomorphisms, for the sake of a unified terminology. We consider $|\cdot|_{L_0}$ the functional defined by:

$$|\cdot|_{L_0} : \begin{cases} \mathcal{D}\mathrm{iff}^r(\mathbb{R}^n) \to \overline{\mathbb{R}^+} \\ f \to |f|_{L_0} := \sup\{|x|^{-1} \cdot |f(x) - x|, x \in \mathbb{R}^n, x \neq 0\} \end{cases}, \tag{2.1}$$

where $|\cdot|$ denotes some norm on $\mathbb{R}^n$. Note that when $f(0) = 0$, $|f|_{L_0}$ represents a bound on the Lipschitz constant of $f - Id_{\mathbb{R}^n}$ w.r.t. 0, provided that this constant does exist; this observation explains our choice of notation.

For all f and g in $\mathrm{Diff}^r(\mathbb{R}^n)$ such that $|f|_{L_0}, |f^{-1}|_{L_0} < \infty$ and $|g|_{L_0}, |g^{-1}|_{L_0} < \infty$, we set:

$$\Delta_0(f, g) := \max(|fg^{-1}|_{L_0}, |g^{-1}f|_{L_0}, |gf^{-1}|_{L_0}, |f^{-1}g|_{L_0}), \tag{2.2}$$

where fg^{-1} stands for $f \circ g^{-1}$.

Remark 1 Note that the conditions $|f|_{L_0}, |f^{-1}|_{L_0} < \infty$ and $|g|_{L_0}, |g^{-1}|_{L_0} < \infty$ imply that $|fg^{-1}|_{L_0}, |g^{-1}f|_{L_0} < \infty$. Indeed, this last assertion results from the inequality:

$$|fg^{-1}|_{L_0} \leq (|f|_{L_0} + |g|_{L_0})(|g^{-1}|_{L_0}(|g|_{L_0} + 1) + 1),$$

and the companion one for $|g^{-1}f|_{L_0}$. The same kind of estimations hold for $|gf^{-1}|_{L_0}, |f^{-1}g|_{L_0}$.

Let $\mathcal{D}^r_{L_0}$ denote the space of diffeomorphisms $f \in \mathcal{C}^r(\mathbb{R}^n)$ such that $|f|_{L_0}, |f^{-1}|_{L_0} < \infty$. In trying to verify whether Δ_0 is a distance, it is easy to see that the separability axiom, as well as the symmetry and positivity ones, are satisfied. Nevertheless, given the definitions (2.1) and (2.2), the triangle inequality does not hold. More precisely, we find that, for each $(f, g, h) \in (\mathcal{D}^r_{L_0})^3$:

$$\Delta_0(f, g) \leq \Delta_0(f, h)\Delta_0(h, g) + \Delta_0(f, h) + \Delta_0(h, g). \tag{2.3}$$

We know from [1] that Δ_0 is a *semi-pseudo metric* of Čech, since the triangle inequality is not satisfied. We call Δ_0 a *sp-metric*, for short; this sp-metric defines a topology, as shown below for Δ_1.

The subgroup $\mathcal{D}^r_{L_0}$ depends on the choice of the functional defined in (2.1). Other choices could be made at this stage, for instance $|f|_{H_0,\beta} := \sup\{|x|^{-\beta} \cdot |f(x) - x|, x \in \mathbb{R}^n, x \neq 0\}$ ($f - Id_{\mathbb{R}^n}$ Hölder w.r.t. 0); such choices would yield other subgroups of $\mathrm{Diff}^r(\mathbb{R}^n)$ for $\beta \neq 1$, by using the same scheme as the one applied to construct $\mathcal{D}^r_{L_0}$.

Remark 2 We show now that the Δ_0-topology is actually finer than the Fréchet $\mathcal{C}^0$-topology induced on $\mathcal{D}^r_{L_0}$. Consider $f, g \in \mathcal{D}^r_{L_0}$. Note that for each compact subset K of $\mathbb{R}^n$, with

$$\Delta_{0,K}(f, g) := \max(|fg^{-1}|_{L_0,K}, |g^{-1}f|_{L_0,K}, |gf^{-1}|_{L_0,K}, |f^{-1}g|_{L_0,K})$$

and $|fg^{-1}|_{L_0,K}$ defined as in (2.1) with the supremum taken on K, we have:

$$|f(x) - g(x)| = |fg^{-1}(g(x)) - g(x)| \leq |g(x)||fg^{-1}|_{L_0,K} \leq C_K\Delta_{0,K}(f, g) \leq C_K\Delta_0(f, g), \forall x \in K, \tag{2.4}$$

where C_K is some positive constant depending on the compact K. Thus each element of the basis of the Fréchet $\mathcal{C}^0$-topology contains an element of the basis of the Δ_0-topology, since the former is defined as a finite intersection of p_K-balls, where p_K is the (supremum) semi-norm affected to K.

The reciprocal is false. Consider a diffeomorphism $f(\neq Id_{\mathbb{R}^n})$ such that $d(f^{n+1}, f^n) \to 0$ as $n \to \infty$, where $d(f, g) = \sum_{j=1}^{+\infty} 2^{-j}p_{K_j}(f, g)/(1 + p_{K_j}(f, g))$ is a possible metric compatible with the Fréchet $\mathcal{C}^0$-topology and $\{K_j\}$ is an exhaustive sequence of compacts of $\mathbb{R}^n$. In any case, we have $\Delta_0(f^{n+1}, f^n) \neq 0$ and thus conclude that the Δ_0-topology is indeed finer than the Fréchet one on $\mathcal{D}^r_{L_0}$. For instance $d(f^{n+1}, f^n) \to 0$ for $f(x) = x/2$ on $\mathbb{R}$, and $\Delta_0(f^{n+1}, f^n) = 1$.

Remark 3 Linear vector fields on $\mathbb{R}^n$ generate diffeomorphisms in $\mathcal{D}^\infty_{L_0}$. Indeed, consider the linear ODE. $\dot{x} = Ax$, where $A \in \mathcal{M}_n(\mathbb{R})$; its flow generates diffeomorphisms with the required smoothness and for each finite t, $|e^{tA}|_{L_0}, |e^{-tA}|_{L_0} < \infty$. This example has motivated in part the presence of $|x|^{-1}$ in definition (2.1). Note also that diffeomorphims generated by linear ODEs are not the only ones concerned: indeed $\mathcal{C}^r$-diffeomorphisms of the form $f(x) = Ax + g(x)$, such that $|f^{-1}|_{L_0} < \infty$, with $|g(x)| \leq C|x|$ or, alternatively, $g(0) = 0$ and $(|g(x)|/|x|) \to 0$ for $|x| \to +\infty$, provide elements of $\mathcal{D}^r_{L_0}$. For instance, on $\mathbb{R}$, if $f(x) = ax + g(x)$, with $g \in \mathcal{C}^r$, g, g' bounded and $|a| > \max|g'(x)|$, then $f \in \mathcal{D}^r_{L_0}$.

In order to prove our persistence theorem, we introduce on each $\mathrm{Diff}^r(\mathbb{R}^n)$ for $r \geq 1$ the mapping $|\cdot|_1$, which is constructed as $|f|_1 := \sup\{\|Df(x) - Id_{\mathbb{R}^n}\|_{\mathcal{L}(\mathbb{R}^n)}, x \in \mathbb{R}^n\}$ for each $f \in \mathrm{Diff}^r(\mathbb{R}^n)$, where $\mathcal{L}(\mathbb{R}^n)$ denotes the usual space of linear mappings on $\mathbb{R}^n$, with the norm $\|\cdot\|$, and define $\mathcal{D}^r_{L_0,1}$ as the set of diffeomorphisms $f \in \mathcal{D}^r_{L_0}$ with $|f|_1, |f^{-1}|_1 < \infty$. Then, for all f and g in $\mathcal{D}^r_{L_0,1}$, we define:

$$\Delta_1(f, g) := \Delta_0(f, g) + \max(|fg^{-1}|_1, |g^{-1}f|_1, |gf^{-1}|_1, |f^{-1}g|_1). \tag{2.5}$$

It is clear that Δ_1 is also a sp-metric for $\mathcal{D}^r_{L_0,1}$, because Δ_0 is. The basis $\mathcal{B}$ of open sets in this sp-metric:

$$\mathcal{B} := \{\mathcal{V}_1(f;\alpha), \alpha \in \mathbb{R}^+, f \in \mathcal{D}^r_{L_0,1}\} \text{ with, for each } \alpha \in \mathbb{R}^+, \mathcal{V}_1(f;\alpha) := \{g \in \mathcal{D}^r_{L_0,1} | \Delta_1(f,g) < \alpha\}, \quad (2.6)$$

generates a topology, since the relaxed triangle inequality (2.3) and the definition of $|\cdot|_1$ permit one to show that, roughly speaking, the intersection of two basis elements contains a third one.

Following now the steps of the analysis in *Remark* 2 for the Δ_1-topology, we can prove

Proposition 2.1 *On the subgroup $\mathcal{D}^1_{L_0,1}(\mathbb{R}^n)$, the Δ_1-topology is finer that the Fréchet $\mathcal{C}^1$-topology.*

3. Persistence of hyperbolic fixed points for suitable Δ_1-perturbations

We call $g \in \mathcal{D}^1_{L_0,1}(\mathbb{R}^n)$ a Δ_1-*perturbation* of $f \in \mathcal{D}^1_{L_0,1}(\mathbb{R}^n)$ if $\Delta_1(f,g)$ is bounded. We are now in a position to state our principal result on persistence of fixed hyperbolic points of a diffeomorphism for sufficiently small Δ_1-perturbations, as well as provide the main elements of the proof.

Theorem 3.1 *Let η be a hyperbolic fixed point of a diffeomorphism $\varphi \in \mathcal{D}^1_{L_0,1}$. There exists a real $\varepsilon > 0$ such that each $\psi \in \mathcal{D}^1_{L_0,1}(\mathbb{R}^n)$ satisfying $\Delta_1(\psi,\varphi) \leq \varepsilon$ has a unique fixed point μ in some neighborhood of η. Furthermore μ is hyperbolic, and the index of μ is equal to the index of η.*

Proof. The proof consists of an application of the implicit function theorem within our framework. More precisely, let us introduce the mapping Φ given by:

$$\Phi : \begin{cases} \mathbb{R}^n \times \mathcal{D}^1_{L_0,1} \to \mathbb{R}^n \\ (\xi,\psi) \to \xi - \psi(\xi) \end{cases} . \quad (3.7)$$

We note that $\Phi(\eta,\varphi) = \varphi(\eta) - \eta = 0$ and let $D_\xi\Phi$ be the derivative of Φ w.r.t. ξ; $D_\xi\Phi(\eta,\varphi) = Id_{\mathbb{R}^n} - D\varphi(\eta)$ is invertible by the hyperbolicity assumption and the finite-dimensionality of $\mathbb{R}^n$. It suffices then to show that Φ and $D_\xi\Phi(\xi,\psi)$ are continuous in (ξ,ψ) in order to apply a weak version of the implicit function theorem, which holds when one factor of the cartesian product constituting the source space is merely topological (e.g., [4]).

We show first the continuity of Φ. Let $((x,f),(y,g)) \in (\mathbb{R}^n \times \mathcal{D}^1_{L_0,1})^2$ and let $|\cdot|$ be any norm in $\mathbb{R}^n$. By our assumptions and definitions, $|g(x) - fg^{-1}(g(x))| \leq |g(x)||fg^{-1}|_{L_0}$; hence

$$|\Phi(x,f) - \Phi(y,g)| \leq |fg^{-1}|_{L_0} \cdot |g(x)| + |g(x) - g(y)| + |x - y|. \quad (3.8)$$

$$|\Phi(x,f) - \Phi(y,g)| \leq \Delta_1(f,g)|g(x)| + |g(x) - g(y)| + |x - y|; \quad (3.9)$$

the continuity of g then yields the continuity of Φ.

We show now the continuity of $D_\xi\Phi$. Because $D_\xi\Phi(x,f) = Id_{\mathbb{R}^n} - Df(x)$, we get naturally

$$\|D_\xi\Phi(x,f) - D_\xi\Phi(y,g)\| = \|Df(x) - Dg(y)\|. \quad (3.10)$$

Furthermore, the chain rule yields $Dg^{-1}(g(x)) = (Dg(x))^{-1}$, thus allowing us to write $Df(x) = Df(x) \circ Dg^{-1}(g(x)) \circ Dg(x)$. By introducing $\Lambda := D(fg^{-1})(g(x)) = Df(x)Dg^{-1}(g(x))$, we have

$$\|D_\xi\Phi(x,f) - D_\xi\Phi(y,g)\| \leq \|\Lambda \circ (Dg(x) - Dg(y))\| + \|(\Lambda - Id_{\mathbb{R}^n}) \circ Dg(y)\| \quad (3.11)$$

and by the definition of Λ and of the norm $|\cdot|_1$,

$$\|D_\xi\Phi(x,f) - D_\xi\Phi(y,g)\| \leq (1 + |fg^{-1}|_1) \cdot \|Dg(x) - Dg(y)\| + |fg^{-1}|_1 \cdot \|Dg(y)\|, \quad (3.12)$$

which proves the continuity of $D_\xi\Phi$ with respect to the Δ_1-topology.

Having shown the required continuity of the map Φ and of its derivative, the above-mentioned implicit function theorem yields the existence of a Δ_1-neighborhood $V(\neq \emptyset)$ of φ in $\mathcal{D}^1_{L_0,1}$, and of a unique continuous map $u : V \to \mathbb{R}^n$, such that $u(\varphi) = \eta$ and $\Phi(u(\psi), \psi) \equiv 0$, for each $\psi \in V$. In this way we get the existence of a fixed point, given by $u(\psi)$, in some neighborhood of η, for each map ψ sufficiently close — in the Δ_1-topology — to the diffeomorphism φ. Furthermore, $D_\xi\Phi(u(\psi), \psi) = I - D\psi(u(\psi))$ and, since $u(\varphi) = \eta$, $D_\xi\Phi(\eta, \varphi) = I - D\varphi(\eta)$ yields

$$\|D_\xi\Phi(u(\psi), \psi) - D_\xi\Phi(\eta, \varphi)\| = \|D\psi(u(\psi)) - D\varphi(\eta)\|. \tag{3.13}$$

To conclude the proof, we recall a basic *structural stability* result on hyperbolic linear isomorphisms of $\mathbb{R}^n$ (e.g., [3]): if A is a hyperbolic linear isomorphism, then there exists a real $\delta(A) > 0$ such that, if $B \in \mathcal{L}(\mathbb{R}^n)$ satisfies $\|B - A\| \leq \delta(A)$, then B is conjugate to A. Therefore, according to (3.13) and the continuity of $D_\xi\Phi$ in the Δ_1-topology, we get, by taking ψ sufficiently close to φ in this topology, that $u(\psi) := \mu$ is a hyperbolic fixed point of ψ with the same index as η. This completes the proof. □

Remark 4 In many physical and biological applications, the dynamics of the system are only known to within a certain margin of error. Let us consider $f(x) = ax$ ($a \in \mathbb{R}^*$, $x \in \mathbb{R}$) and $g(x) = (a + \varepsilon)x$. Then f and g are ε-close in both the Fréchet $\mathcal{C}^1$-topology and the Δ_1-topology, e.g. $\Delta_1(f, g) = \mathcal{O}(\varepsilon)$ is small. According to Theorem 3.1, persistence of hyperbolic fixed points holds for (some) diffeomorphisms of $\mathbb{R}^n$, when the error in the system's parameters is sufficiently small. It is at least suggested, and remains to be verified, that the allowable error in the framework adopted here may be larger than the one permitted by a version of Theorem 3.1 for the Fréchet $\mathcal{C}^1$-topology.

Remark 5 Note that a diffeomorphism $f \in \mathcal{D}^r_{L_0}$ satisfies necessarily $f(0) = 0$, because $|f|_{L_0}$ is finite. It is easy, though, to extend this sp-metric in order to deal with diffeomorphisms for which $f(0) \neq 0$, cf. the following remark.

Remark 6 Note that our main result still holds by endowing $\mathrm{Diff}^r(\mathbb{R}^n)$ with the natural metric $\Delta(f, g) := \max(|\mathrm{fg}^{-1}|_{\delta_0}, |\mathrm{g}^{-1}\mathrm{f}|_{\delta_0}, |\mathrm{gf}^{-1}|_{\delta_0}, |\mathrm{f}^{-1}\mathrm{g}|_{\delta_0})$ deduced from the functional $|\cdot|_{\delta_0}$; this functional is defined for each $f \in \mathcal{D}\mathrm{iff}^r(\mathbb{R}^n)$ by $|f|_{\delta_0} := \sup\{|f(x) - x|, x \in \mathbb{R}^n\}$.

The proof in this case follows the same outline, and is actually more straightforward. The metric $|f|_{\delta_0}$ evaluates the gap between f and the identity map, at order 0. This metric permits one to take into account diffeomorphisms with $f(0) \neq 0$, as well as those for which $f(0) = 0$. To deal solely with the latter, one needs to build a sp-metric of Čech, rather than a metric. We have adopted the latter framework for the presentation of our results, because the persistence results are less easy to obtain in this topological context, and because many important diffeomorphisms, such as the linear ones, cannot be handled by the simpler, metric approach.

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