A Note on Existence of Random Attractors

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We present here two approaches, one based on Lyapunov-type method (Sections 1 and 2) and the second one based on the use of energy estimates combined with Langevin equation for obtaining pullback dissipation (Section 3).

1 A Lyapunov-type method

We focus on the following set of SDEs, in the Stratonovich interpretation:

$$dx_i = f_i(x)dt + \sigma_i(x) \circ dW_t, \; \forall i \in \{1, ..., d\}, \tag{1}$$

where $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, $d \in \mathbb{N}^*$ and W is the standard $\mathbb{R}$-valued Wiener process[1]. For every $i \in \{1, ..., d\}$, we assume that $f_i \in C_b^{1,\delta}(\mathbb{R}^d)$ and $\sigma_i \in C_b^{2,\delta}(\mathbb{R}^d)$ for $\delta \in (0, 1]$, that insure that (1) is well-posed for convenient initial data and that the solutions define a *cocycle* (*cf.* [4] for the notations and the existence of solutions, and *cf.* [1] for the existence of the cocycle). Part of the method exposed here is based on the work [5], but is presented in this notes within the framework of stochastic differential equations, whereas the latter deals essentially with the random differential equations (RDEs) setting.

We denote by $\langle \cdot, \cdot \rangle$ the canonic inner product of $\mathbb{R}^d$, and by $|| \cdot ||$ its associated norm. A d-vector-valued function will be written in bold character when a latin letter will be used. For instance $\mathbf{f}$, will denote the map $x \mapsto (f_1(x), \cdots, f_d(x))$, but σ will denote the map $\sigma : x \mapsto (\sigma_1(x), \cdots, \sigma_d(x))$.

We assume that there exists a map $V : \mathbb{R}^d \to \mathbb{R}$, such that $V \in C^1(\mathbb{R}^d, \mathbb{R}^+)$, verifying the following conditions,

$$\exists \, \alpha, \beta > 0 \; : \; \langle \mathbf{f}(x), \nabla V(x) \rangle + \alpha \cdot V(x) \leq \beta, \; \forall \, x \in \mathbb{R}^d, \tag{2}$$

and such that[2]

$$\exists \, \gamma \in \mathbb{R} \; : \; \langle \sigma(x), \nabla V(x) \rangle = \gamma \cdot V(x), \; \forall \, x \in \mathbb{R}^d. \tag{3}$$

This last condition will serve as a cross condition between the *diffusion non-linearity*, σ, and the *drift non-linearity*, $\mathbf{f}$, as we will see in applications (*cf.* next section).

Central to our approach is the use of a smoothed approximation of the Wiener process. In that respect we proceed as follows. Introduce $\phi_\epsilon(t) = \epsilon^{-1}\phi(t/\epsilon)$, for a nonnegative $\mathbb{R}$-valued function $\phi \in C^1(\mathbb{R})$ satisfying supp$\phi \subset [0, 1]$, and $\int_0^\epsilon \phi(t)dt = 1$. Introduce further $W_t^\epsilon(\omega) := \int_{-\infty}^{\infty} \phi_\epsilon(\tau - t)W_\tau(\omega)d\tau = \int_0^\epsilon \phi_\epsilon(u)W_{u+t}(\omega)du$. Compute for $h \neq 0$, and $u, t \in \mathbb{R}$,

$$\phi_\epsilon(u+h)W_{u+h+t}(\omega) - \phi_\epsilon(u)W_{u+t}(\omega) = (\phi_\epsilon(u+h) - \phi_\epsilon(u))W_{u+h+t}(\omega) + \phi_\epsilon(u)(W_{u+h+t}(\omega) - W_{u+t}(\omega)),$$

now using

[1] The case of diffusion part of type $\sum_{j=1}^m \sigma_{ij}(x) \circ dW_t^j$ can be handled via the method presented here. We make this simplification only for a better understanding of the arguments, see Remark 1.4

[2] Note that if $\sigma_i = 0$, for every i, then this condition leads to the following estimate $V(x(t, x)) \leq V(x)e^{-\alpha t} + \frac{\beta}{\alpha}(1 - e^{-\alpha t}), \forall t > 0$; for $x(t, x)$ denoting the solution of $dx = \mathbf{f}(x)dt, x(0) = x$. Then every bounded set B of $\mathbb{R}^d$ is absorbed in the set $\{x \in \mathbb{R} : V(x) \leq 2\frac{\beta}{\alpha}\}$, insuring the existence of a global attractor for this flow. The the idea consists then to preserve this property for appropriate random perturbations.

$$W_{u+t}(\omega) = W_u(\theta_t\omega) + W_t(\omega),\ \forall u,\ \forall t \in \mathbb{R},$$

by the definition of the *metric dynamical system* θ canonically associated to the Wiener process (*cf.* [1], or the slides); we get by making $h \to 0$ and integrating between 0 and ϵ,

$$\frac{d}{dt}W_t^\epsilon(\omega) = -\int_0^\epsilon \dot{\phi}_\epsilon(u)W_u(\theta_t\omega)du, \tag{4}$$

that we can rewritten as

$$\frac{d}{dt}W_t^\epsilon(\omega) = \rho_\epsilon(\theta_t\omega), \tag{5}$$

where

$$\rho_\epsilon(\omega) := -\frac{1}{\epsilon^2}\int_0^\epsilon \dot{\phi}(\frac{\tau}{\epsilon})W_\tau(\omega)d\tau. \tag{6}$$

The resulting constructed $\mathbb{R}$-valued random variables $\{\rho_\epsilon(\omega)\}_{\omega\in\Omega}$ serve to build a family of RDEs that generate a family of cocycle parametrized by ϵ that approximate the cocycle of original family of SDEs (1) in an appropriate way (*cf.* section Appendix A for details). For that we consider the following set of RDEs,

$$\dot{x_i} = f_i(x) + \sigma_i(x)\rho_\epsilon(\theta_t\omega),\ \forall\ i \in \{1, ..., d\}. \tag{7}$$

We can then use these RDEs to obtain estimates thanks to classical calculus.

Let $\varphi^\epsilon(t,\omega)$ be the corresponding cocycle and $x^\epsilon(t) := x^\epsilon(t;x) = \varphi^\epsilon(t,\omega)x$ be the solution of (7) passing through x at $t = 0$. Let us lastly denote by $V_\epsilon(t)$ the map $t \mapsto V(x^\epsilon(t))$, then we have that:

$$\frac{dV_\epsilon}{dt} = \langle\nabla V(x^\epsilon(t)), \frac{dx^\epsilon(t)}{dt}\rangle,$$

that is,

$$\frac{dV_\epsilon(t)}{dt} = \langle\nabla V(x^\epsilon(t)), \mathbf{f}(x)\rangle + \langle\nabla V(x), \rho_\epsilon(\theta_t\omega)\cdot\sigma(x)\rangle, \tag{8}$$

where we have used the following notation for every $\omega \in \Omega$,

$$\rho_\epsilon(\omega)\cdot\sigma(x) := (\rho_\epsilon(\omega)\cdot\sigma_1(x_1), \cdot\cdot\cdot, \rho_\epsilon(\omega)\cdot\sigma_d(x_d)).$$

Now by (2) and (3), we get

$$\frac{dV_\epsilon(t)}{dt} \leq (-\alpha + \gamma\cdot\rho_\epsilon(\theta_t\omega))V_\epsilon(t) + \beta.$$

Therefore,

$$V_\epsilon(t) \leq V(x)\exp\Big\{-\alpha t + \gamma\int_0^t \rho_\epsilon(\theta_\tau\omega)d\tau\Big\} + \beta\cdot\int_0^t \Big\{\exp(-\alpha(t-s) + \gamma\int_s^t \rho_\epsilon(\theta_\tau\omega)d\tau)\Big\}ds,$$

that is

$$V(\varphi^\epsilon(t,\theta_{-t}\omega)x) \leq V(x)\exp\{-\alpha t + \gamma\int_{-t}^0 \rho_\epsilon(\theta_u\omega)du\} + \beta\cdot\int_{-t}^0\{\exp(\alpha s + \gamma\int_s^0 \rho_\epsilon(\theta_u\omega)du)\}ds, \tag{9}$$

by using the *one-parameter group property of* $\{\theta_t\}_{t\in\mathbb{R}}$ and the change of variables $u = \tau - t$ in the corresponding integrals; and then operating the substitution $(s-t) \leftarrow s$ in the resulting last integral.

Now because of (5), we obtain $\int_s^0 \rho_\epsilon(\theta_\tau\omega) = -W_s^\epsilon(\omega) = \int_0^\epsilon \dot{\phi}_\epsilon(\tau) W_s(\theta_\tau\omega)d\tau$, and since $\tau \mapsto W_s(\theta_\tau\omega)$ is *continuous* for every s and ω we get,

$$\lim_{\epsilon\to 0}\int_s^0 \rho_\epsilon(\theta_\tau\omega)d\tau = -W_s(\omega). \tag{10}$$

Now using a *Wong-Zakai type theorem* (*cf.* Appendix A) we obtain that:

$$\exists\, \Omega^* \subset \Omega,\ \theta\text{-invariant set } : \mathbb{P}(\Omega^*) = 1, \tag{11}$$

for which

$$\exists\, \epsilon_k \to 0 \text{ s.t. for any } x \in \mathbb{R}^d \text{ and } \omega \in \Omega^*,\ \varphi^{\epsilon_k}(t,\theta_{-t}\omega)x \to \varphi(t,\theta_{-t}\omega)x, \text{ for almost all } t > 0; \tag{12}$$

where φ denotes the cocycle associated to (1).

Therefore by passing to the limit in (9), with the sequence $\{\epsilon_k\}$ just defined above, we obtain using (10):

$$V(\varphi(t,\theta_{-t}\omega)x) \le V(x)\exp\Big\{-\alpha t - \gamma W_{-t}(\omega)\Big\} + \beta\int_{-t}^0 \exp\Big\{\alpha s - \gamma W_s(\omega)\Big\}ds, \tag{13}$$

for almost all $t > 0$, and for all $\omega \in \Omega^*$, $x \in \mathbb{R}^d$. [3]

Let us consider the random variable $R(\omega) := \int_{-\infty}^0 \exp(\alpha s - \gamma W_s(\omega))ds$, then we can show that for every ω this integral exists. Indeed we know from the law of large number[4] that almost surely,

$$\lim_{t\to +\backslash -\infty}\frac{W_t(\omega)}{t} = 0, \tag{14}$$

and hence for any $\epsilon > 0$ and a.a. $\omega \in \Omega$ there exists a random constant $0 < c(\omega) < \infty$ s.t.

$$|W_t(\omega)| \le \epsilon|t| + c(\omega),\ \forall t \in \mathbb{R}, \tag{15}$$

leading to the existence of $R(\omega)$.

Define now the *random set*[5] B through,

$$B(\omega) := \{x \in \mathbb{R}^d : V(x) \le 2\frac{\beta}{\alpha} + 2R(\omega)\}.$$

The estimate (13) leads then to,

$$V(\varphi(t,\theta_{-t}\omega)x) \le V(x)\exp\Big\{-\alpha t - \gamma W_{-t}(\omega)\Big\} + R(\omega), \tag{16}$$

for every $\omega \in \Omega^*$, $x \in \mathbb{R}^d$, and $t > 0$.

[3] We can in fact obtain this inequality for all $t > 0$, by the continuity of $t \mapsto \varphi(t,\theta_{-t}\omega)x$.

[4] the Wiener process is i.i.d. with mean 0

[5] The precise definition of a random set and the proof that B is a random set is given in step 2 of the proof of Theorem 1.2

Thanks to (15), by choosing ϵ sufficiently small (and so $c(\omega)$ large), we obtain that there exists a random constant $M(\omega)$ such that $V(\varphi(t,\theta_{-t}\omega)x) \leq V(x)M(\omega)e^{(-\alpha+\epsilon)t} + R(\omega)$, for every $\omega \in \Omega^*$, $x \in \mathbb{R}^d$, and $t > 0$. Thus, for every deterministic bounded set L of $\mathbb{R}^d$, because V is continuous, $V(x)$ is bounded on L (more exactly on its closure) and therefore there exists $t_0(\omega, L)$ such that for every $t \geq t_0(\omega, L)$,

$$\varphi(t,\theta_{-t}\omega)L \subset B(\omega).$$

At this stage of the reasoning we have almost concluded that the random set B is absorbing every deterministic bounded set of $\mathbb{R}^d$ for the RDS (θ, φ). Indeed, the last inclusion is valid, by construction on Ω^*, of full measure but still distinct of Ω. To extend the absorbing property over the whole Ω, we can redefine φ for every $\omega \in \Omega - \Omega^*$ by the formula $\varphi(t,\omega)x := y(t,x)$ where $y(t,x)$ denotes the solution of $\dot{y} = \mathbf{f}(y)$ passing through x at $t = 0$. Then thanks to (2), we get for every $\omega \in \Omega - \Omega^*$,

$$V(\varphi(t,\theta_{-t}\omega)x) \leq V(x)e^{-\alpha t} + \frac{\beta}{\alpha}(1 - e^{-\alpha t}), \forall t > 0.$$

Then following the same arguments, it is not difficult to conclude that for any $\omega \in \Omega - \Omega^*$ and any deterministic bounded set L, there exists $t'_0(\omega, L)$ such that for every $t \geq t'_0(\omega, L)$, $\varphi(t,\theta_{-t}\omega)L \subset B(\omega)$. We have thus shown that B is absorbing every deterministic bounded set of $\mathbb{R}^d$ for the RDS (θ, φ). We can use then the Theorem of [2] to conclude the existence of a random attractor, we have thus proved the following theorem:

Theorem 1.1 *Consider the following set of SDEs in the Stratonovich interpretation:*

$$dx_i = f_i(x)dt + \sigma_i(x) \circ dW_t, \ \forall i \in \{1, ..., d\}, \tag{17}$$

where $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ *and,* $d \in \mathbb{Z}^+$. *For every* $i \in \{1, ..., d\}$, *we assume that* $f_i \in C_b^{1,\delta}(\mathbb{R}^d)$ *and* $\sigma_i \in C_b^{2,\delta}(\mathbb{R}^d)$, *for* $\delta \in (0, 1]$. *Assume furthermore that there exists a map* $V : \mathbb{R}^d \to \mathbb{R}^+$, *such that* $V \in C^1(\mathbb{R}^d, \mathbb{R}^+)$, *verifying the following conditions :*

(i) $\exists\, \alpha, \beta > 0 \ : \ \langle \mathbf{f}(x), \nabla V(x) \rangle + \alpha \cdot V(x) \leq \beta, \ \forall\, x \in \mathbb{R}^d$,

(ii) $\exists\, \gamma \in \mathbb{R} \ : \ \langle \sigma(x), \nabla V(x) \rangle = \gamma \cdot V(x), \ \forall\, x \in \mathbb{R}^d$.

Then there exists a random attractor $\{\mathcal{A}(\omega)\}_{\omega \in \Omega}$ *pullback attracting every deterministic bounded sets of* $\mathbb{R}^d$.

In fact, we can improve this result. Assume now that there exist some constants $\delta, b, m > 0$ such that

$$V(x) \leq b||x||^\delta + m, \text{ for all } x \in \mathbb{R}^d, \tag{18}$$

then we can show that there exists a random attractor attracting the *tempered* compact random sets (*cf.* Definition 3.2, for a definition of *tempered sets*). More precisely we prove the following theorem

Theorem 1.2 *Under the assumptions of Theorem 1.1, and condition (18), then the system (1) possess a random attractor pullback attracting the tempered compact random sets.*

Remark 1.3 *We can in fact show that the random attractor provided by Theorem 1.1 and Theorem 1.2 are exactly the same, in set theoretic meaning, the last theorem consists therefore of an improvement of the former, since it says that the random attractor attracts more objects than the deterministic bounded sets.*

Proof of Theorem 1.2. We proceed in several steps.

Step1. First we prove that $R(\omega)$ is a tempered random variable w.r.t. θ canonically associated to the Wiener process. Recall that

$$R(\omega) := \int_{-\infty}^{0} \exp(\alpha s - \gamma W_s(\omega))ds,$$

so $R(\theta_t\omega) = \int_{-\infty}^{0} \exp(\alpha s - \gamma W_s(\theta_t\omega)ds$. Besides for every $\epsilon > 0$, there exists $0 < c(\omega) < \infty$ s.t. $|W_s(\theta_t\omega)| \leq |W_{s+t}(\omega)| + |W_t(\omega)| \leq 2\epsilon|t+s| + c(\omega)$, and therefore for every $t \in \mathbb{R}$ and almost all ω we have that there exists a random constant $M(\omega)$ s.t.:

$$R(\theta_t\omega) \leq \frac{M(\omega)}{\alpha - 2\epsilon} e^{2\epsilon|t|},$$

for every $\epsilon > 0$ s.t. $\alpha - 2\epsilon > 0$. Now for $k > 0$ if we select $\epsilon > 0$ such that $\epsilon < \min(\alpha/2, k/2)$ then $\sup_{t\in\mathbb{R}}(e^{-k|t|}R(\theta_t\omega))$ is finite. We have thus proved that $R(\omega)$ is tempered.

Step 2. Consider now the ω-parameterized family of sets $B(\omega) := \{x \in \mathbb{R}^d : V(x) \leq 2\frac{\beta}{\alpha} + 2R(\omega)\}$. This family is random *closed tempered set*. We prove only here that it is a *random set*[6], since the temperedness is obvious from the Definition 3.2 and the fact that $R(\omega)$ is a tempered random variable. It is clear that $B(\omega)$ is closed for any $\omega \in \Omega$. Due to the equivalent definitions of random sets (*cf.* Arnold and references therein) it is sufficient to show that $\{\omega : B(\omega) \cap U = \emptyset\}$ is measurable for every open sets $U \subset \mathbb{R}^d$, to conclude on the measurability of B. Hereafter we focus on $B'(\omega) = \{x \in \mathbb{R}^d : V(x) \leq R(\omega)\}$, on which it is enough to show the measurability. Observe that

$$\mathbb{R}^d \backslash B'(\omega) = V^{-1}(\mathbb{R}) \backslash V^{-1}((-\infty, R(\omega)) = V^{-1}(R(\omega), +\infty)).$$

$U \subset \mathbb{R}^d \backslash B(\omega)$ if and only if $V(U) \subset V^{-1}(R(\omega), +\infty))$, that implies

$$\{\omega : B'(\omega) \cap U = \emptyset\} = \{\omega : R(\omega) < u, \text{ for any } u \in V(U)\},$$

and therefore

$$\{\omega : U \subset \mathbb{R}^d \backslash B'(\omega)\} = \bigcap_{n\in\mathbb{N}} \{\omega : R(\omega) < u_n\},$$

where $u_n \in V(U)$ and $u_n \to \inf V(U)$ when $n \to +\infty$. That concludes that B is a random set.

Step3. We consider the universe $\mathcal{D}$ of all random compact tempered sets of $\mathbb{R}^d$. Using the estimate (16) and the condition (18) we can now show that if $x(\omega) \in D(\omega)$ for an arbitrary $D \in \mathcal{D}$, then:

$$V(\varphi(t, \theta_{-t}\omega)x(\theta_{-t}\omega)) \leq (b\|x(\theta_{-t}\omega)\|^\delta + m)e^{-\alpha t - \gamma W_{-t}(\omega)} + R(\omega),$$

and because $x(\omega) \in D(\omega)$, $x(\omega)$ is a random $\mathbb{R}^d$-valued tempered variable (*cf.* again Definition 3.2), that allows to conclude that $B(\omega)$ is absorbing in the universe $\mathcal{D}$. Indeed using the temperedness and preceding arguments about the Wiener, we can show that there exists a random constant $C(x,\omega) > 0$ (depending on $x \in D$), a random constant $0 < M(\omega) < \infty$ and $k, \epsilon > 0$ such that $\delta k + \epsilon - \alpha < 0$ and:

$$V(\varphi(t, \theta_{-t}\omega)x(\theta_{-t}\omega)) \leq b(C(x,\omega))^\delta M(\omega)e^{(\delta k+\epsilon-\alpha)t} + mM(\omega)e^{(\epsilon-\alpha)t} + R(\omega), \ \forall t > 0,$$

leading to $\varphi(t, \theta_{-t}\omega)x(\theta_{-t}\omega) \in B(\omega)$ for t sufficiently big. Now using the fact that $C(x,\omega)$ is a constant satisfying $e^{-k|t|}\|x(\theta_t\omega)\| \leq C(x,\omega)$ for all $t \in \mathbb{R}$, we get that if x varies in D a random compact set, then $C(\cdot,\omega)$ is bounded on D. We obtain therefore that there exists $t_0(\omega, D)$ such that

$$t \geq t_0(\omega, D) \Rightarrow \varphi(t, \theta_{-t}\omega)x(\theta_{-t}\omega) \in B(\omega), \quad \text{for every } x \in D.$$

[6] A random set is a family $B(\omega)_{\omega\in\Omega}$ of subsets of $\mathbb{R}^d$, such that its graph $B := \{(\omega, x) : x \in \mathbb{R}^d\}$ is product measurable. We identify random sets with their graphs.

This last statement gives the precise meaning of being absorbing in the universe $\mathcal{D}$.

We have thus shown by using theorem 3.11 of [2] that, under the supplementary condition (18), there exists a random attractor $\{A(\omega)\}$ pullback attracting all the random tempered compact sets of $\mathbb{R}^d$. The proof of theorem (1.2) is complete. $\square$

Remark 1.4 *The arguments developed here are valid for general diffusion part such that* $\sum_{j=1}^{j=m} \sigma_{ij}(x) \circ dW_t^j$*, by regularizing each Wiener* $t \mapsto W_t^j$*; conditioned of course to the existence of a functional* V *satisfying (2) and (3).*

2 An application to a general class of SDEs

We now apply the preceding strategy on a particular class of SDEs, to illustrate the usefulness of this strategy. We can extend further the class that I propose here only to illustrate. To simplify we assume that $\mathbf{f}$ and σ are smooth on $\mathbb{R}^d$ [7].

We consider now the following structural assumptions on $\mathbf{f}$ and σ. For the drift part we assume that there exits a smooth $\mathbb{R}^d$-valued function $\mathbf{g}$ s.t.

$$\exists \alpha > 0 \ : \ \forall\, i \in \{1, ..., d\}, \ f_i(x) \leq -\alpha x_i + g_i(x), \ \forall x \in \mathbb{R}^d \tag{19}$$

and for the diffusion part we assume the following conditions:

$$\forall\, i \in \{1, ..., d\}, \ \forall x \in \mathbb{R}^d, \ \sigma_i(x) = \sigma_i(x_i), \tag{20}$$

$$\forall\, i \in \{1, ..., d\}, \ \sigma_i(s) \neq 0, \ \text{for every } s \in \mathbb{R}, \tag{21}$$

and

$$\forall\, i \in \{1, ..., d\}, \ \exists \mu_i > 0, \ \exists M_i > 0, \ \frac{s}{\sigma_i(s)} \geq \mu_i, \ \text{for } |s| \geq M_i, \tag{22}$$

we supplement these conditions by the cross condition:

$$\exists M > 0 \ : \ \left\| \frac{g_i(x)}{\sigma_i(x)} \right\| \leq M, \ \text{ for all } x \in \mathbb{R}^d, \tag{23}$$

Lastly assume that

$$\alpha \cdot \inf\{\mu_i \ , \ i \in \{1, ..., d\}\} - M > 0. \tag{24}$$

We show now that under these conditions we are able to obtain a functional V that satisfy (2) and (3). Set for every i,

$$V_i(u) = \exp\left\{\delta \int_1^u \frac{ds}{\sigma_i(s)}\right\}, \ \text{for every } u \in \mathbb{R},$$

where δ is a parameter to be adjusted later and then consider

$$V(x) := \sum_{i=1}^{i=d} V_i(x_i).$$

[7] we can assume without loss of generality that σ is smooth almost everywhere

Then V is C^1 and V satisfies (3) with $\gamma = \delta$.
We have for every $x \in \mathbb{R}^d$,

$$\langle \mathbf{f}(x), \nabla V(x) \rangle = \delta \sum_{i=1}^{i=d} V_i(x_i) \cdot \frac{f_i(x)}{\sigma_i(x_i)} \leq -\alpha\delta \sum_{i=1}^{i=d} \frac{x_i}{\sigma_i(x_i)} \cdot V_i(x_i) + \delta \sum_{i=1}^{i=d} V_i(x_i) \frac{g_i(x)}{\sigma_i(x_i)}.$$

Besides, there exists a constant $C > 0$ such that for every $x_i \in \mathbb{R}$, $\frac{x_i}{\sigma_i(x_i)} \cdot V_i(x_i) \geq \mu V_i(x_i) - C$, with $\mu := \inf\{\mu_i \ , \ i \in \{1, ..., d\}\}$.

Therefore,

$$\langle \mathbf{f}(x), \nabla V(x) \rangle \leq -\alpha\delta\mu \cdot V(x) + \delta M V(x) + C,$$

that permits to fit the assumptions of Theorem 1.1, because of assumption (24). With have thus the existence of a random attractor attracting deterministic bounded sets.

Here is a short list of SDEs, for $d = 1$, that fulfill the conditions, and for which therefore it exists a random attractor. Hereafter $\alpha > 0$ and can be adjusted if needed.

- $$dx = (-\alpha x + |x|^r)dt + (\frac{x}{|x|}\sqrt{|x| + 1}) \circ dW_t, \text{ for any } 0 < r < 1/2.$$
- $$dx = -\alpha x \exp(x^r)dt + \big(\frac{x}{|x|} \exp(-|x|^s)\big) \circ dW_t, \quad \text{for any } r, s > 0$$
- $$dx = -\alpha x dt + \big(\frac{x}{|x|} \ln(|x|^r + 2)\big) \circ dW_t, \text{ for any } r > 0$$
- $$dx = \big(-\alpha x + (1 + |x|)^{p/q}\big) + \frac{x}{|x|} \cdot \big(\sigma|x| + \epsilon\big) \circ dW_t, \text{ for any integers } 0 < p < q, \text{ and any } \epsilon, \sigma > 0,$$
- $$dx = -\alpha x dt + \frac{x}{|x|}(\tanh(\lambda x) + 2) \circ dW_t, \text{ for any } \lambda > 0$$

Remark 2.1 *The conditions reported here are not optimal, the goal was only to illustrate that the procedure developed in section 1 gives the existence of random attractors in non-trivial situations, even in dimension =1.*

Remark 2.2 *Note that for certain of SDEs depicted above, we obtain the existence of random attractor for a class of noise which is not* **multiplicative** *w.r.t. to the classical definition, i.e. the stochastic equation does not possesses as deterministic fixed points, the equilibrium points of the deterministic part. Of course, with these examples, we are far from the "pure" additive case, ϵdW_t, which requires less sophisticated method w.r.t. to the existence of random attractor. Of course this method can handle the case of certain (nonlinear) multiplicative noises.*

3 Existence of random attractors in the stochastic Lorenz system and Galerkin approximation of Navier-Stokes

Comments: This part is intended to present the key elements related to a proof of existence of random attractors. The ideas are presented here in finite dimensions, for a class of SDEs that arise in Galerkin approximation of stochastic Navier-Stokes equations, or in the stochastic Lorenz system. The material presented here is based on the work of Crauel and Flandoli [2]. The main originality of the present discussion relies maybe on its presentation.

Let H be a finite dimensional Hilbert space. We consider A a positive definite symmetric matrix acting on H such that $\langle Ax, x\rangle \geq |x|^2$, where $\langle \cdot, \cdot \rangle$ denotes the inner product on H and $|\cdot|$ its related norm[8]. We consider $B : H \times H \to H$ a bilinear mapping with the property:

$$(25) \qquad \langle B(y, x), z\rangle = -\langle B(y, z), x\rangle, \text{ for every } x, y, z \in H.$$

Finally, $\{W_t\}$ will denote the two-sided Wiener process with values in H. Within this framework, we will focus on SDE of the form:

$$(26) \qquad dX_t = (-AX_t - B(X_t, X_t) + f)dt + \gamma dW_t,$$

with $\gamma, f \in H$. Note that we will consider only the case γ =constant for the sake of clarity. The method used here can be extended, with more technicality, for more general mutiplicative noise (see concluding comments at the end of this part).

The Eq. (26) leads to the following pathwise formulation:

$$(27) \qquad X_t^{t_0,x}(\omega) = x + \int_{t0}^{t} (-AX_s^{t_0,x}(\omega) - B(X_s^{t_0,x}(\omega), X_s^{t_0,x}(\omega)) + f)ds + \gamma(W_t(\omega) - W_{t_0}(\omega)),$$

where $X_t^{t_0,x}(\omega)$ denotes the value at time t of the solution passing through x at time t_0, for a fixed realization ω. Note that this equation can be solved for every continuous path $t \mapsto W_t(\omega)$ by a classical contraction principle (see e.g. [1, 4]).

In order to match with the general theory of random dynamical systems, we need to introduce some notations. However we will avoid to use the "jargon" of this theory, by simply defining the stochastic flow as [1, 4]:

$$\varphi(t, \omega)x := X_t^{0,x}(\omega),$$

and defining a time parametrization $\{\theta_t\}_{t\in\mathbb{R}}$ of Ω (the probability space) through the relation:

$$\varphi(t, \theta_{-t_0}\omega)x := X_{t-t_0}^{-t_0,x}(\omega).$$

As in the deterministic case [7], a key element in the proof of existence of a random attractor is the existence of a bounded absorbing set which is in this case random[9]. More precisely we want to prove the following theorem.

Theorem. For $\mathbb{P}$-a.e. ω, there exits a random bounded ball $B(0, r(\omega))$ such that for every bounded set $B \subset H$, there exists a time $t_{\omega,B}$ s.t.:

$$\varphi(t, \theta_{-t}w)B \subset B(0, r(\omega)), \quad \forall t \geq t_{\omega,B}.$$

Proof. Since $\varphi(t, \theta_{-t_0}\omega)x := X_{t-t_0}^{-t_0,x}(\omega)$ it suffices to prove that for every bounded subset B of H, $\sup_{t_0\geq 0}\sup_{x\in B}|X_0^{-t_0,x}(\omega)| < \infty$, $\mathbb{P}$-a.s, with $\mathbb{P}$ the probability on Ω.

Step 1, a change of variable[10]. Consider for every $t_0 < t$, the following linear pathwise integral version of an equation of Langevin type:

$$Z_t(\omega) = Z_{t_0}(\omega) - \int_{t_0}^{t} AZ_s(\omega)ds + \gamma(W_t(\omega) - W_{t_0}(\omega)).$$

[8] We can also consider $\langle Ax, x\rangle \geq K|x|^2$, with $K > 0$, we have chosen $K = 1$ only for simplifying the presentation.

[9] A random bounded closed set is a map $B : \Omega \to 2^H$ taking values in the bounded closed sets of H, which is *measurable* in the sense that for each $x \in H$ the map $\omega \mapsto d(\omega, B(\omega))$ is measurable.

[10] The use of a change of variable will be clarified in the details of the proof (see also step 4.)

Let us consider the difference between X_t and Z_t, i.e.,

$$Y_t^{t_0,x}(\omega) = X_t^{t_0,x}(\omega) - Z_t(\omega),$$

then,

$$Y_t^{t_0,x}(\omega) = x - Z_{t_0}(\omega) - \int_{t_0}^{t} AY_s^{t_0,x}(\omega)ds + \int_{t0}^{t}(-B(X_s^{t_0,x}(\omega), X_s^{t_0,x}(\omega)) + f)ds,$$

that we can write as the following random differential equation (RDE) driven by the process Z,

$$\frac{dY_t^{t_0,x}(\omega)}{dt} + AY_t^{t_0,x}(\omega) = -B(Y_t^{t_0,x}(\omega) + Z_t(\omega), Y_t^{t_0,x}(\omega) + Z_t(\omega)) + f.$$

Now we are in position to get an estimate of $|Y_0^{-t_0,x}(\omega)|^2$, for x varying in a bounded subset B of H, and $t_0 \leq 0$. First we obtain a key inequality satisfied by $Y_t^{t_0,x}(\omega)$.

Step 2, an inequality for dissipativity. To make less cumbersome the computations, we drop some notations:

$$\frac{dY}{dt} + AY = -B(Y + Z, Y + Z) + f.$$

Consider the energy:

$$\frac{1}{2}\frac{d|Y|^2}{dt} = \langle Y, \frac{dY}{dt}\rangle = -\langle Y, AY\rangle + \langle Y, -B(Y + Z, Y + Z) + f\rangle,$$

from $\langle Ax, x\rangle \geq |x|^2$, and the properties of B we get,

$$\frac{1}{2}\frac{d|Y|^2}{dt} + |Y|^2 \leq -\langle Y, B(Y + Z, Y)\rangle - \langle Y, B(Y + Z, Z)\rangle + \langle Y, f\rangle = -\langle Y, B(Y + Z, Z)\rangle + \langle Y, f\rangle,$$

and so,

$$\frac{1}{2}\frac{d|Y|^2}{dt} + |Y|^2 \leq C|Y||Z|(|Y| + |Z|) + |Y||f| = C|Y|^2|Z| + C|Y|(|Z|^2 + |f|),$$

which leads to

$$\frac{1}{2}\frac{d|Y|^2}{dt} + |Y|^2 \leq C|Y|^2|Z| + \epsilon|Y|^2 + \frac{C}{\epsilon}[|Z|^2 + |f|]^2, \tag{28}$$

by a Young inequality. Now the main task is to use (28) in order to get an appropriate estimate $|Y|^2$ that characterizes a form of dissipativity. At this stage there is two main ingredients, one is conceptual and use the pullback approach for apprehending dissipativity in random dynamical systems, the other is technical and concerns the term $C|Y|^2|Z|$.

Step3, the advantage of a pullback approach. At this stage of reasoning we suppress the term $C|Y|^2|Z|$, in order to avoid technicalities and to present the main ideas. The way of extending the present step, with the term $C|Y|^2|Z|$, will be discussed in the next step.

If we assume $C|Y|^2|Z|$ absent, then we have

$$\frac{1}{2}\frac{d|Y|^2}{dt} + |Y|^2 \leq \epsilon|Y|^2 + \frac{C}{\epsilon}[|Z|^2 + |f|]^2,$$

that we can write

$$\frac{1}{2}\frac{d|Y|^2}{dt} + \nu|Y|^2 \leq C[|Z|^2 + |f|]^2,$$

for some $\nu > 0$ and $C > 0$, a new constant.

By a version of the Gronwall lemma, we get:

$$|Y_t^{t_0,x}(\omega)|^2 \leq |x|^2 e^{-\nu(t-t_0)} + \int_{t_0}^t e^{-\nu(t-s)} C[|Z_s(\omega)|^2 + |f|]^2 ds, \tag{29}$$

where we came back to the full notations for the need of the following discussion. We want to show the advantage of the pullback approach for dealing with (28) w.r.t. a more classical push-forward approach. In the last one, we get on a interval $[0,t]$:

$$|Y_t^{0,x}(\omega)|^2 \leq |x|^2 e^{-\nu t} + \int_0^t e^{-\nu(t-s)} C[|Z_s(\omega)|^2 + |f|]^2 ds.$$

Therefore if $C[|Z_s(\omega)|^2 + |f|]^2$ is very big, lets say greater than a constant M over an interval $[t-1,t]$, then

$$\int_0^t e^{-\nu(t-s)} C[|Z_s(\omega)|^2 + |f|]^2 ds \geq \int_{t-1}^t e^{-\nu(t-s)} M ds = M \frac{1-e^{-\nu}}{\nu},$$

and the fluctuation of $Z_s(\omega)$ are immediately experienced at time t by the system.

Let us look on what happen with a pullback vision. If we work pull-back on $[-t,0]$, we have by (29)

$$|Y_0^{-t,x}(\omega)|^2 \leq |x|^2 e^{-\nu t} + \int_{-t}^0 e^{\nu s} C[|Z_s(\omega)|^2 + |f|]^2 ds. \tag{30}$$

If $C[|Z_s(\omega)|^2 + |f|]^2 \sim M$ over an interval $[-\tau-1,-\tau]$ ($\tau > 0$), then $\int_{-\tau-1}^{-\tau} e^{\nu s} C[|Z_s(\omega)|^2 + |f|]^2 ds$ contributes to the integral $\int_{-t}^0 e^{\nu s} C[|Z_s(\omega)|^2 + |f|]^2 ds$ with the following magnitude:

$$\int_{-\tau-1}^{-\tau} e^{\nu s} C[|Z_s(\omega)|^2 + |f|]^2 ds \sim M \frac{e^{-\nu\tau} - e^{-\nu\tau}}{\nu} = e^{-\nu\tau} M \frac{1-e^{-\nu}}{\nu},$$

and therefore this integral over a time frame equals to one, is roughly $e^{-\nu\tau}$-times smaller than in the forward case. The interpretation is that the fluctuations of $Z_s(\omega)$ at some distant time $-\tau$ in the past (from time 0) are experienced by the system but at that time, and they are damped later.

We are now in position to conclude in the simple case, i.e. where the term $C|Y|^2|Z|$ is absent. From (30), we have that:

$$\sup_{t_0 \geq 0} \sup_{x \in B} |Y_0^{-t_0,x}(\omega)|^2 \leq \sup_{t_0 \geq 0} \sup_{x \in B} |x|^2 e^{-\nu t_0} + \sup_{t_0 \geq 0} \int_{-t_0}^0 e^{\nu s} C[|Z_s(\omega)|^2 + |f|]^2 ds,$$

which leads to

$$\sup_{t_0 \geq 0} \sup_{x \in B} |Y_0^{-t_0,x}(\omega)|^2 \leq C_B + C(\omega),$$

with $C(\omega) = \int_{-\infty}^0 e^{\nu s} C[|Z_s(\omega)|^2 + |f|]^2 ds$.

Now because, $\lim_{t \to +/-\infty} (\frac{Z_t(\omega)}{t^\beta}) = 0$, $\mathbb{P}$-a.s. if $\beta > 0$ is appropriately chosen (see e.g. [1, 2])[11], we have $C(\omega) < \infty$, $\mathbb{P}$-a.s.; that concludes that $\sup_{t_0 \geq 0} \sup_{x \in B} |Y_0^{-t_0,x}(\omega)|^2 < \infty$, which in turn gives,

$$\sup_{t_0 \geq 0} \sup_{x \in B} |X_0^{-t_0,x}(\omega)|^2 < \infty,$$

that is deduced, recall it, without the term $C|Y|^2|Z|$, in the inequality (28).

[11] This last equality can be obtained through the Borel-Cantelli lemma. We see here the importance of using this auxiliary variable in the change of variable $Y = X - Z$. The property of Z (together with the pullback approach), allows to conclude on a dissipativity property of Y, and in return of X.

Pullback absorption, in presence of the term $C|Y|^2|Z|$. We go back to the original problem, that is to show the pullback absorption in presence of the term $C|Y|^2|Z|$.

The main inequality is now given by:

$$\frac{1}{2}\frac{d|Y|^2}{dt}+(\nu-C|Z|)|Y|^2\leq +C[|Z|^2+|f|]^2,$$

and the term $e^{-\nu(t-t_0)}$ that appeared in the preceding computations is now replaced by the term $\exp(-\int_{t_0}^{t}(\nu-C|Z_s(\omega)|)ds)$, which does tend towards 0 as $t-t_0$ tends to infinity (because $\lim_{(t-t_0)\to\infty}(\frac{-1}{t-t_0}\int_{t_0}^{t}(\nu-C|Z_s(\omega)|)ds)=-\nu+C\mathbb{E}(Z(\cdot))$, that could be >0). To overcome this difficulty we consider the following damped Langevin equation, for $\alpha>0$:

$$dZ_{t,\alpha}+(A+\alpha I)Z_{t,\alpha}dt=\gamma dW_t,$$

and denote by $Z_{t,\alpha}$ its solution.

Now by introducing the change of variables, $X=Y-Z_\alpha$ (omitting some notations) and reproducing the preceding computations, we obtain that Y satisfies the inequality:

$$\frac{1}{2}\frac{d|Y|^2}{dt}+|Y|^2\leq C|Y|^2|Z_\alpha|+\epsilon|Y|^2+\frac{C}{\epsilon}[|Z_\alpha|^2+\alpha|Z_\alpha|+|f|]^2,$$

which leads to

$$\frac{1}{2}\frac{d|Y|^2}{dt}+(\nu-C|Z_\alpha|)|Y|^2\leq C[|Z_\alpha|^2+\alpha|Z_\alpha|+|f|]^2.$$

The main interest of introducing this damped version of the variable Z introduced in step 1, is that

$$\lim_{(t-t_0)\to\infty}\frac{-1}{t-t_0}\int_{t_0}^{t}(\nu-C|Z_{s,\alpha}(\omega)|)ds=-\nu+C\mathbb{E}(Z_\alpha(\cdot))$$

and $\mathbb{E}(Z_\alpha(\cdot))$ can be rendered arbitrarily small, for α sufficiently large. Therefore there exists an $\alpha>0$ such that,

$$-\nu+C\mathbb{E}(Z_\alpha(\cdot))\leq\frac{-\nu}{2},$$

and we finally get,

$$\exp(-\int_{t_0}^{t}(\nu-C|Z_{s,\alpha}(\omega)|)ds)\sim\exp(-\frac{\nu}{2}(t-t_0)),$$

as $(t-t_0)$ tends to infinity.

We can now go back to the rest of the proof performed in step 3, to conclude that $\mathbb{P}$-a.s,

$$\sup_{t_0\geq 0}\sup_{x\in B}|X_0^{-t_0,x}(\omega)|<\infty.$$

The proof is complete [12]. □

Concluding comments.

- When this step of showing a random bounded set absorbing bounded subsets of H is achieved, the existence of a random attractor for the stochastic flow φ is obtain from the general result of Theorem 3.11 of [2], which is in some sense an extension in a random setting of the Theorem 1.1 of [7].

[12] Here we see the usefulness of an appropriately chosen auxiliary Z to perform a change of variables. In the new variable, Y, exhibits a good dissipativity property and because the property of Z, we can go back to the original variable, X.

- The proof presented here, have been performed in the case of the so-called additive noise. The case of (Stratonovich) multiplicative noise, $X_t \circ dW_t$ e.g., can be treated along the same lines, and is detailed in [2] for instance. For more general multiplicative noise, the change of variables needed involves more sophisticated algebra as presented in [6], which still allows to prove existence theorems of random attractors in theory, but still need more investigations for practical purpose. This will be a nest topic of investigation.

Appendix A: A Wong-Zakai type theorem

We consider the Stratonovich SDEs

$$dx_i = f_i(x)dt + \sum_{j=1}^{m} \sigma_{ij}(x) \circ dW_t^j, \forall\, i \in \{1, ..., d\}, \tag{31}$$

where $f_i \in C_b^{1,\delta}(\mathbb{R}^d)$ and $\sigma_{ij} \in C_b^{2,\delta}(\mathbb{R}^d)$, for some $\delta \in (0,1]$. Then following (Kunita, Chap 3) we have that this system of Stratonovich SDEs has a unique solution $x(t, \omega; x_0)$ for every initial data $x_0 \in L^2(\Omega, \mathcal{F}, \mathbb{P})$. We can then state a Wong-Zakai type theorem, based on *e.g* [4] or theorem 6.8.2 of [3] (with references therein).

Theorem 3.1 *Let $x(t, \omega; x_0)$ be the solution to the Stratonovich SDEs (1) with initial data $x_0 \in \mathbb{R}^d$ and $x^\epsilon(t, \omega; x_0)$ be the solution to the RDEs (7). Then*

$$\lim_{\epsilon \to 0} \mathbb{E}\Big\{ \sup_{[0,T]} \sup_{\|x_0\| \leq R} \|x(t; x_0) - x^\epsilon(t; x_0)\|^2 \Big\} = 0,$$

for any $T > 0$ and $R > 0$.

In terms of cocycles we can show[13] that this theorem implies that there exists a sequence $\epsilon_k \to 0$ such that the set Ω^* of all $\omega \in \Omega$ satisfying

$$\lim_{n \to +\infty} \int_a^b \Big\{ \sup_{[0,T]} \sup_{\|x_0\| \leq R} \|\varphi(t, \theta_s\omega)x_0 - \varphi^{\epsilon_n}(t, \theta_s\omega)x_0\|^2 \Big\} ds = 0,$$

for all $a < b$, $T > 0$ and $R > 0$; has a full measure, that is satisfies $\mathbb{P}(\Omega^*) = 1$, and is θ-invariant. In this last identity, φ denotes the cocycle generated by (1) and φ^{ϵ_n} the one generated by (7).

Appendix B: Complements on some RDS tools

In this section we consider a general RDS (φ, θ), acting on a metric phase space X.

Definition 3.2 *A random set $\{D(\omega)\}_\omega$ is said to be tempered with respect to a metric dynamical system $\theta := (\Omega, \mathcal{F}, \mathbb{P}, \{theta_t\}_{t \in \mathbb{R}}$, if there exists a random variable $r(\omega)$ and an element $y \in X$ s.t.:*

$$D(\omega) \subset \overline{B(y, r(\omega))}, \text{ for all } \omega \in \Omega,$$

where $B(y, r(\omega))$ denotes the open ball centered at y and of radius $r(\omega)$, and where $r(\omega)$ is a tempered random variable w.r.t. θ, i.e.

$$\sup_{t \in \mathbb{R}} \big\{ e^{-k|t|} |r(\theta_t\omega)| \big\} < \infty, \text{ for all } \omega \in \Omega, \text{ and for all } k > 0.$$

A random variable $v(\omega)$ with values in X is said to be tempered if the one-point random set $\{v(\omega)\}$ is tempered.

According to this definition every deterministic bounded set is tempered.

[13] I would developed it in a forthcoming note

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